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Using this book

Contents of the Chapters

Opening pages to sections:
- These give an overview of the content covered by the section.
- Trigger question
- An exercise for thinking about a problem
- A problem connected to the main text

Main text:
- Trigger question
- Practical example for understanding the content
  - This question is a clue to solving the problem.
- A problem to check if you have correctly understood the basic content
- A problem to make clearer your understanding of the content
- Problem for which a calculator is useful
- Further description relating to the term in a footnote

Let's try!
- Extra problems to help you think about the content

At the end of each chapter...

Write in your exercise books:
- The useful techniques you have learned
- What you found interesting
- What you learned and enjoyed doing
# Problems

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<td>These cover subject matter you need to master.</td>
<td>This shows where a topic is covered in the main text.</td>
<td>These problems review the whole chapter and show applications of what you have learned. These are grouped by difficulty into A and B* problems.</td>
<td>These sections explore topics relating to what you studied in the chapter, in further depth.</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>These sections present topics of interest connected to what you are studying.</td>
</tr>
</tbody>
</table>

# Diversions*

These are puzzles which the math you have learned will help you to solve.

# Appendix*

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<th>Let’s Research</th>
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</tr>
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# Development

The following are further development topics.

- Similar Terms: 55
- Projections and Sections: 170
- The World of Finite Number Systems: 174
- Representing Inequalities: 178
- Incircle and Circumcircle of a Triangle: 180

The sections labeled "Development" are not part of the formal curriculum requirements, but are provided to widen the scope of study.
Chapter 1 Positive and Negative Numbers

What location has the largest difference between minimum and maximum air temperatures?

For one day in March:

<table>
<thead>
<tr>
<th>Location</th>
<th>Maximum air temperature</th>
<th>Minimum air temperature</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kanazawa</td>
<td>-3</td>
<td>-8</td>
</tr>
<tr>
<td>Kochi</td>
<td>11</td>
<td>3</td>
</tr>
<tr>
<td>Naha</td>
<td>20</td>
<td>14</td>
</tr>
</tbody>
</table>

For which location is the minimum air temperature lowest?
Hirosaki

9

2

Yokohama

How many degrees are there between the minimum and maximum air temperatures for Kutchan?
Following the example shown for Yokohama, color in the thermometers to represent the minimum and maximum air temperatures, and try to see the answer.

The maximum air temperature for Kutchan is shown as "-3°C". What does the sign "-" mean?

Do the same investigation for the other locations.

1 — Positive and Negative Numbers
Signed Numbers

Taking 0°C as a reference point, temperatures lower than this are written using a minus sign ("-"). For example, the temperature 3 degrees lower than 0°C is -3°C, read as "negative three degrees (Celsius)."

Temperatures above 0°C may also be written with a plus sign ("+"), as in +5°C. This is read as "positive five degrees (Celsius)."

When plus and minus signs are used in this way, they are also called "positive sign" and "negative sign."

Problem 1 Write the following temperatures, using plus and minus signs.

1. The temperature 5.5 degrees below 0°C
2. The temperature 8 degrees above 0°C

Numbers such as +5 and +8 are called positive numbers, and numbers such as -3 and -5.5 are called negative numbers. The number zero is neither positive nor negative.

The numbers +5 and +8 are the same as the numbers 5 and 8 that you already know. We will now think about numbers as including negative numbers, and to make this clear, we will refer to them as "signed numbers." For example, the integers (whole numbers) consist of the positive integers, zero, and the negative integers.

The positive integers are also known as the natural numbers.

Integers

... -3, -2, -1, 0, 1, 2, 3, ...

Negative integers Positive integers (natural numbers)

Amounts representing opposite qualities can be shown as positive and negative numbers.

1 — Positive and Negative Numbers
If we indicate 500 yen income as +500 yen, then −300 yen represents 300 yen expenditure.

If we take sea level as zero, then we can write heights above sea level with a plus sign, so that the 3776-meter height of Mount Fuji appears as +3776 m.

Suruga Bay with Mount Fuji (Shizuoka)

Check 1 Following Example 2, express the following heights using plus and minus signs.

1. Nobeyama Station ... 1345.67 m above sea level
2. Yoshioka Undersea Station ... 145 m below sea level

Yoshioka Undersea Station (Hokkaido) Nobeyama Station (Nagano)

Problem 2 The elevation of Lake Biwa is 85 meters, or in other words, the surface of Lake Biwa is 85 meters above sea level.
If the elevation of the Caspian Sea is −28 m, what does this tell us about the relationship of the surface of the Caspian and sea level.
If we represent a movement of 5 meters to the east from point A as +5 m, then we can represent a movement of 3 meters to the west from A as −3 m.

**Problem 3**

In Example 3, what movements would +10 m and −2 m represent?

**Example 4**

The following map shows forecast maximum air temperatures for various places in Japan, and the numbers in parenthesis show the differences from the previous day. For example, for Sendai, (−2) indicates that the forecast maximum air temperature for the day is 2 degrees lower than the maximum air temperature for the previous day.

So what was the maximum temperature in Sendai the previous day?

**Problem 4**

In Example 4, Fukuoka is shown as (+2) and Sapporo as (0). What do these figures mean?

**Problem 5**

Express the heights of the following mountains relative to the height of Yarigadake, 2903 m, using a plus sign to represent a higher value and a minus sign for a lower value.

- Shakushidake 2812 m
- Shiroumadake 2932 m

Yarigadake, Shakushidake, and Shiroumadake (Nagano)
Comparing Numbers

The number line

On the number line shown on the right, show the positions of 2 and 3.5.

Now we’ll make a number line to include negative numbers.

First, take a reference point on a straight line to represent the number zero. Next, make markings at fixed intervals to left and right on the line, and mark the positive numbers to the right of zero and the negative numbers to the left of zero, as in the following figure.

Check 1 Name the numbers that correspond to points A, B, and C on the line above.

The point on the number line corresponding to zero is called the origin.

The direction to the right on the number line is called the positive direction, and the direction to the left is called the negative direction.

Check 2 On the number line below, mark the positions corresponding to the following numbers.

1  2  3  4

1 — Positive and Negative Numbers
Comparing numbers

Within the positive numbers, numbers increase in value as we move to the right. Similarly, when the number line includes zero and the negative numbers, we say numbers increase to the right, and decrease to the left.

For example, considering −1 and +3, then −1 is to the left on the number line, and we say that −1 is less than +3. We can express this as follows:

\[-1 < +3\]

The comparison symbols < and > are called inequality signs.

**Example 1** Comparing −1 and −3

On the number line, −1 is to the right of −3, and therefore −1 is greater:

\[-3 < -1\]

Comparing 0, −3 and +2

On the number line, 0 is to the right of −3, and +2 is to the right of 0, therefore:

\[-3 < 0, 0 < +2\]

We can combine these expressions in the following way:

\[-3 < 0 < +2 \quad \text{or} \quad +2 > 0 > -3\]

Positive numbers are greater than zero; negative numbers are less than zero.

**Check 3** Use inequality signs to show a comparison of the following sets of numbers.

1. 1, −3, 5
2. 0, +1, −2

**Problem 1** Use inequality signs to show a comparison of the following sets of numbers.

1. −0.1, −1
2. −1/2, −2
3. −6, +3, −8
Absolute values

For the numbers +5, −5, and −2.3, what are the distances of the corresponding points on the number line from the origin?

On the number line, the distance from the origin to the point corresponding to a certain number is called the absolute value of that number. For example, the distance from the origin to +3 is 3, and so the absolute value of +3 is 3. Similarly, −3 is at a distance 3 from the origin, so the absolute value of −3 is also 3. The absolute value of zero is 0.

Check 4 Give the absolute values of the following numbers:

1. +8
2. −10
3. +2.5
4. −1/3

Problem 2 Give the numbers whose absolute value is 7.

Positive numbers increase as the absolute value increases. But what happens with negative numbers?

If we compare −11 and −15, for example, −15 has a larger absolute value than −11, but −15 is to the left of −11 on the number line. Therefore:

−15 < −11

Negative numbers decrease as the absolute value gets larger.

Problem 3 Use inequality signs to show a comparison of the following sets of numbers.

1. −36, −49
2. −0.8, −0.12
Basic Exercises

Amounts representing opposite qualities
Fill in the blanks in the following statements with appropriate words or numbers.

1. If we represent 8 minutes after the present as +8 minutes, then -5 minutes represents the present.

2. If we represent the weight of an object being 6 kg heavier than a reference object as +6 kg, then we can represent being 10 kg lighter as kg.

Positive and negative numbers and the number line
On the number line below, mark the positions corresponding to the following numbers 1 to 4. Also give the numbers corresponding to points A, B, C, and D.

Comparing numbers
Use inequality signs to show a comparison of the following sets of numbers.

1. +3, -4
2. -13, -8
3. +6, -9, 0

Absolute values
Which of the following set of numbers have the same absolute values?

-3, 0.3, 0, -1, +3, +\frac{1}{3}, -0.3
Addition and Subtraction

1 Addition

Looking at a scene with a straight road extending east-west, we'll consider how to add signed numbers.

If a girl walks as shown in the cases below, starting from point A, what would be the result? To get the same result, how far and in which direction should she walk from point A?

1. First walk 3 m to the east, then walk 5 m to the east.
2. First walk 3 m to the east, then walk 5 m to the west.

We will indicate a movement to the east with a positive number, and a movement to the west with a negative number.

We'll express the result of the two movements in case (1) above, using signed numbers.

The first movement is +3 m.
The second movement is +5 m.

The figure on the right shows the effect of these two movements happening one after the other, and the result is +8 m.

The result +8 m in Example 1 can be shown by the following expression:

\[(+3) + (-5) = 8\]
Example 2
We’ll express the result of the two movements in case (2) in the \( \Rightarrow \) on the previous page, using signed numbers.

The first movement is \(+3\) m.
The second movement is \(-5\) m.

The figure on the right shows the effect of these two movements happening one after the other, and the result is \(-2\) m.

The result \(-2\) m in Example 2 can be shown by the following expression:

\[
(+3) + (-5) = -2
\]

The result of addition is called the sum of the numbers.

Let’s investigate how to calculate sums of signed numbers.

Example 3
Adding numbers with the same sign
1. Sum of +4 and +6
   
   \[
   (+4) + (+6) = +10
   \]

2. Sum of \(-4\) and \(-6\)

   \[
   (-4) + (-6) = -10
   \]

Check 1
Calculate the following.
1. \((+2) \cdot (-7)\)
2. \((-2) + (-4)\)
Adding numbers with the same absolute value and opposite signs

Sum of $-5$ and $+5$

\[
\begin{array}{c}
\text{Sum of} \quad -5 \quad \text{and} \quad +5 \\
\hspace{1cm} \begin{array}{c}
\text{Add} \\
\text{Subtract} \\
\text{Add}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
-5 \\
+5
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
-5 \\
+5
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
-5 \\
+5
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
= 0
\end{array}
\end{array}
\]

The sum of two numbers with the same absolute value and opposite signs is zero.

Adding numbers with opposite signs

1. Sum of $+9$ and $-4$

\[
\begin{array}{c}
\begin{array}{c}
+9 \\
-4
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
+5 \\
+4
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
-10 \\
+4
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
-6
\end{array}
\end{array}
\]

2. Sum of $-10$ and $+4$

\[
\begin{array}{c}
\begin{array}{c}
-4 \\
+4
\end{array}
\end{array}
\]

Check 2 Calculate the following.

\[
\begin{array}{c}
\begin{array}{c}
( + 4) + ( - 3)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
( + 7) + ( - 9)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
( - 6) + ( + 6)
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
( - 12) + ( - 18)
\end{array}
\end{array}
\]

Here is how to find the sum of two numbers:

1. When the numbers have the same sign
   Apply the same sign to the sum of the absolute values.

\[
( - 4) + ( - 6) = - (4 + 6)
\]

Add

2. When the numbers have opposite signs
   Subtract the smaller absolute value from the larger absolute value, and apply the sign of the number with the larger absolute value.

\[
( - 10) + ( + 4) = - (10 - 4)
\]

Subtract
And here is how addition involving zero works.

If we add zero to any number, the sum is the same as the original number.

\[( - 3 ) + 0 = - 3 \]

If we add any number to zero, the sum is the same as the added number.

\[0 + ( - 5 ) = - 5 \]

**Problem 1**

Calculate the following.

1. \((-3) + (+8)\)
2. \((-5) + (-7)\)
3. \(0 + (-2)\)
4. \((-4) + (-4)\)
5. \((+15) + (-8)\)
6. \((-20) + (+12)\)
7. \((-19) + (-13)\)
8. \((+24) + (-36)\)

Let's look at addition with fractions and decimals.

**Example 6**

1. \((-0.8) + (-1.5)\)
   
   \[= -(0.8 + 1.5)\]
   
   \[= -2.3\]

2. \((-\frac{2}{3}) + (-\frac{1}{4})\)
   
   \[= -(\frac{8}{12} + \frac{3}{12})\]
   
   \[= -(\frac{8 - 3}{12})\]
   
   \[= -\frac{5}{12}\]

Simply calculate in the same way as with integers.

**Problem 2**

Calculate the following.

1. \((+4.8) + (-5.2)\)
2. \((-3) + (+2.6)\)
3. \((-\frac{1}{5}) + (-\frac{3}{5})\)
4. \((+\frac{3}{8}) + (-\frac{5}{8})\)
5. \((+\frac{1}{2}) + (-\frac{3}{7})\)
6. \((-\frac{3}{5}) + (+\frac{1}{10})\)
Commutative law and associative law of addition

When we add one positive number to another, it makes no difference to the sum if we interchange the numbers. The same holds true of negative numbers. This is called the **commutative law of addition**.

\[(+5)+(-3)=(-3)+(+5)\]

---

**Tigger**

Calculate the value of the following expressions \(a\) and \(b\), and compare the results.

\(a\) \[\{(+3)+(-9)\}+(+7)\]

\(b\) \[\{(+3)+((-9)+(+7))\}\]

The following rule also holds true of signed numbers. This is called the **associative law of addition**.

\[\text{\(\boxed{+\text{box}}\)}+(\text{\(\boxed{+\text{box}}\)})\text{\(\boxed{\triangle}\)}=\text{\(\boxed{+\text{box}}\)}+(\text{\(\boxed{+\text{box}}\)})\text{\(\boxed{\triangle}\)}\]

Since the commutative and associative laws hold for addition, when adding signed numbers, you can rearrange the order or grouping without changing the result.

\[\begin{align*}
1 & \quad (+3)+(-8)+(+7)+(-5) \\
& = (+3)+(+7)+(-8)+(-5) \\
& = \{(+3)+(+7)\}+\{(-8)+(-5)\} \\
& = (+10)+(-13) \\
& = -3
\end{align*}\]

\[\begin{align*}
2 & \quad (+6)+(-18)+(-6) \\
& = (+6)+(-6)+(-18) \\
& = \{(+6)+(-6)\}+(-18) \\
& = 0+(-18) \\
& = -18
\end{align*}\]

**Problem**

Calculate the following.

\[\begin{align*}
1 & \quad (+5)+(-9)+(-7)+(+6) \\
2 & \quad (-8)+(+5)+(-3)+(+8)+(-1)
\end{align*}\]
Subtraction

What number fits in the box?

$$\square + (+5) = +8 \quad \ldots \quad (1)$$

The following subtraction finds the number that fits in the box in equation (1).

$$(+8) - (+5) = \quad$$

Also, as you can see from the figure on the right, the number that fits in the box can also be found using addition as follows:

$$(+8) + (-5) = \quad$$

Therefore,

$$(-8) - (+5) = (+8) + (-5)$$

Which means that we can convert subtraction into addition.

What number fits in the box?

$$\square + (+5) = +2 \quad \ldots \quad (2)$$

The calculation to find the number that fits in the box in equation (2) can also be shown as a subtraction, as follows:

$$(-2) - (+5) = \quad$$

Problem 1)

We can also convert the subtraction above to an addition:

$$(-2) + (-5) = \quad$$

Check this by drawing an extra arrow in the figure on the right.
What number fits in the box?

The calculation to find the number that fits in the box in equation (3) can also be shown as a subtraction, as follows:

\[(+2) - (-5) = \square\]

Again, as you can see from the figure on the right, the number that fits in the box can also be shown as an addition, as follows:

\[(+2) + (+5) = \square\]

Thus in this case too, a subtraction can be converted to an addition.

\[(+2) - (-5) = (+2) + (+5)\]

The result of subtraction is called the difference of the numbers.

**Problem 2** Convert the following subtraction expressions into additions.

1. \((+7) - (+5)\)
2. \((+3) - (-5)\)

We can summarize the calculation of differences of signed numbers as follows.

**Subtraction of signed numbers**

Subtracting a signed number is equivalent to adding the number with the sign reversed.

**Example 1**

1. \((+3) - (+7)\)
   
   \[= (+3) + (-7)\]
   
   \[= -4\]

2. \((-3) - (-8)\)
   
   \[= (-3) + (+8)\]
   
   \[= +5\]

**Check 1** Calculate the following.

1. \((-2) - (+9)\)
2. \((-8) - (-4)\)
As is clear in Example 2, the result of subtracting a number from zero is the same as reversing the sign of the number.

Subtracting zero from any number leaves a difference equal to the original number.

\[( -3) - 0 = -3\]

**Problem 3** Calculate the following.

1. \((-6) - (-1)\)
2. \((+7) - (-9)\)
3. \((-3) - (+5)\)
4. \((+9) - (-3)\)
5. \((-5) - (-5)\)
6. \((-7) - (+7)\)
7. \(0 - (-8)\)
8. \((-14) - 0\)

**Problem 4** Calculate the following.

1. \((-0.4) - (+0.2)\)
2. \((-\frac{1}{2}) - (-\frac{2}{3})\)

**Problem 5** The following numbers indicate the locations of stations, based on Okayama at the origin, with positive values to the east, and negative values to the west.

<table>
<thead>
<tr>
<th>Location</th>
<th>Distance</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hakata</td>
<td>-442.0 km</td>
</tr>
<tr>
<td>Hiroshima</td>
<td>-161.3 km</td>
</tr>
<tr>
<td>Okayama</td>
<td>0 km</td>
</tr>
<tr>
<td>Shin-Osaka</td>
<td>+180.3 km</td>
</tr>
</tbody>
</table>

What would these values be if expressed with Hiroshima as the origin. Use signed numbers to express this.
Calculation Involving Both Addition and Subtraction

In elementary school we only considered positive numbers and zero, so that there was no answer to a problem such as 6 - 9, involving subtracting a larger number from a smaller one. However, by including negative numbers, we can do calculation like this.

\[ 6 - 9 = (+6) - (+9) = (+6) + (-9) = -3 \]

Since 6 - 9 is the same as (+6) + (-9), we can think of the sum of +6 and −9.

We can handle expressions including both addition and subtraction in the same way. For example:

\[ 4 - 7 + 9 - 5 \]

If we convert this expression to pure addition, we get the following:

\[ 4 - 7 + 9 - 5 = (+4) - (+7) + (+9) - (+5) = (+4) + (-7) + (+9) + (-5) \]

Thus the expression (1) represents the sum of the following numbers:

\[ +4, \quad -7, \quad +9, \quad -5 \]

These numbers are called the terms of expression (1).

Check 1 Name the terms of the following expressions.

1. \( -6 + 2 - 7 \)
2. \( 2 - 3 - 6 \)

We can think of expression (1) above as being the expression (2) with the parentheses and addition signs deleted, thus being simply a list of the terms. We also omit a plus sign at the beginning of the expression.

Problem 1 Rewrite each of the following expressions as a list of terms, as in expression (1) above.

1. \( (-3) + (+8) + (-4) \)
2. \( (-5) (-2) + 3 \)
Using the commutative law and associative law of addition, we can calculate expression (1) on the previous page as follows:

\[
\begin{align*}
4 - 7 + 9 - 5 & = 4 + 9 - 7 - 5 \\
& = 13 - 12 \\
& = 1 \\
4 - 7 + 9 - 5 & = 4 + 9 - 7 - 5 \\
& = 13 - 12 \\
& = 1
\end{align*}
\]

**Check 2** Calculate the following.

1. \(-4 + 12 - 9\)
2. \(-5 + 3 - 2 + 6\)

**Example 1** Calculate \(-17 - \(-25\) + 3 + \(-14\).**

**Answer**

**Check 3** Calculate the following.

1. \(9 + \(-3\) - \(-5\)\)
2. \(7 + \(-6\) - 4 + \(-9\)\)

**Problem 2** Calculate the following.

1. \(2 - 8 + 7 - 2 + 3\)
2. \(1 - 10 + 13 - 15\)
3. \(-17 - \(-28\) + 0 - 19\)
4. \(12 - 18 - \(-21\) - 11\)

**Problem 3** Calculate the following.

1. \(1.3 - 2.4 - 0.5\)
2. \(5.3 + \(-6.1\) - \(-3.4\)\)
3. \(1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{3}\)
4. \(-\frac{5}{6} - \left(\frac{3}{4}\right) + \frac{1}{2}\)

---

22 1 — Positive and Negative Numbers
Basic Exercises

Addition

Calculate the following.

1. \((-4) + (+5)\)
2. \((+2) + (-18)\)
3. \((+9) + (-9)\)
4. \((-13) + (-7)\)
5. \((-12) + (+3)\)
6. \((-9) + (-6)\)

Subtraction

Calculate the following.

1. \((-9) - (-2)\)
2. \((+7) - (-6)\)
3. \((+2) - (+10)\)
4. \((-4) - (-4)\)
5. \((-4) - (-11)\)
6. \((-15) - (+7)\)

Combined addition and subtraction

Calculate the following.

1. \(-2 + 9 - 6\)
2. \(5 - 6 - 8 - 1\)
3. \(-1 - (-3) - 5\)
4. \(10 + (-15) - (-13) - 23\)

Window on math ... Number brickwork

The sum of the two numbers in two adjacent bricks is written in the brick above them.

In the figures below, fill in all of the numbers to complete the brickwork.

1. \((-7)\)
2. \(-5\)
3. \(0\)
4. \(5\)
5. \(2\)
6. \(4\)
7. \(3\)
8. \(6\)
9. \(-3\)
Multiplication

We can also consider multiplication of signed numbers by thinking about movement east and west.

If we walk at 4 km per hour for 2 hours, how many kilometers have we walked?

We will represent movement to the east by positive numbers, and movement to the west by negative numbers.

First let's consider the case of walking to the east at 4 km per hour.

Problem 1 In the following cases ① and ②, where has the person walked to?

Show the movement from the current position using signed numbers.

① Two hours after the current time
② Two hours before the current time

The result of question ① in Problem 1 is +8 km, which we can find from the following expression:

\[(+4) \times (+2) = +8\]

For question ②, we write the multiplication to find the result.

Since 2 hours before is equivalent to -2 hours after, the expression is as follows:

\[(+4) \times (-2) = -8\]
Next, let's consider the case of walking to the west at 4 km per hour.

**Problem 2**

In the following cases 1 and 2, where has the person walked to? Show the movement from the current position using signed numbers.

1. Two hours after the current time
2. Two hours before the current time

<table>
<thead>
<tr>
<th>2 hours after</th>
<th>Current time</th>
<th>2 hours before</th>
</tr>
</thead>
<tbody>
<tr>
<td>West</td>
<td>4 km/hour</td>
<td>East</td>
</tr>
<tr>
<td></td>
<td>4 km/hour</td>
<td></td>
</tr>
</tbody>
</table>

We take walking to the west at 4 km per hour to be walking at a speed of -4 km per hour, and then we can express the calculations required for questions 1 and 2 of Problem 2 as the following multiplications:

In the case 1: \((-4) \times (2) = -8\)
In the case 2: \((-4) \times (-2) = +8\)

The result of multiplication is called the product of the numbers.

**Problem 3**

When we multiply two numbers with the same sign, what is the sign of the product? And what if we multiply two numbers with opposite signs?

We can summarize the calculation of products of signed numbers as follows.

**Multiplication of signed numbers**

To find the product of two signed numbers, proceed as follows.

1. If the numbers have the same sign, add a plus sign to the product of the absolute values.
2. If the numbers have different signs, add a minus sign to the product of the absolute values.
Multiplying numbers with the same sign
1. \((-3) \times (+4) = -(3 \times 4)\) 
   \(= -12\)
2. \((-3) \times (-4) = +(3 \times 4)\) 
   \(= 12\)

Check 1
Calculate the following.
1. \((+2) \times (+3)\)
2. \((-6) \times (-5)\)

Multiplying numbers with different signs
1. \((-3) \times (-4) = -(3 \times 4)\) 
   \(= -12\)
2. \((-3) \times (+4) = -(3 \times 4)\) 
   \(= -12\)

Check 2
Calculate the following.
1. \((+5) \times (-7)\)
2. \((-6) \times (+2)\)

Problem 4
Calculate the following.
1. \((-8) \times (-3)\)
2. \((+8) \times (+10)\)
3. \((+7) \times (-8)\)
4. \((-9) \times (+7)\)
5. \((-7) \times (+2)\)
6. \((-11) \times (-6)\)
7. \((+15) \times (+4)\)
8. \((+17) \times (-2)\)

Problem 5
1. \((-2.5) \times (-0.6)\)
2. \((-\frac{3}{4}) \times (-\frac{4}{9})\)
Multiply each of the following numbers by $-1$, and compare the result with the original number.

1. $1 + 3$
2. $2 - 3$
3. $3 + 5$
4. $4 - 5$

Multiplying a number by $-1$ has the same effect as reversing the sign.

$$\begin{align*}
(-1) \times 6 &= (-1) \times ( + 6) = -6
\end{align*}$$

Thus $-6$ is equal to $(-1) \times 6$.

$-(-3)$ is equal to $(-1) \times (-3)$.

$$\begin{align*}
1 &-(-3) = (-1) \times (-3) \\
2 &- (-1) = 1
\end{align*}$$

Problem 6 Following the method of Example 4, simplify $-(-5)$ and $-(+2)$.

Problem 7 If we compare 6 and 8 using an inequality sign, the result is $6 < 8$.

Use an inequality sign to show a comparison of $(-1) \times 6$ and $(-1) \times 8$.

Multiplying any number by one results in the original number.

Multiplying one by any number results in the number you multiplied by.

Multiplying any number by zero, or multiplying zero by any number, always results in zero.

Problem 8 For east–west movement as we considered on pages 24 and 25, what do the following relations mean:

$$\begin{align*}
\bullet \times 0 &= 0 \\
0 \times \bullet &= 0
\end{align*}$$
Commutative law and associative law of multiplication

When we multiply one positive number by another, it makes no difference to the product if we interchange the numbers. The same holds true of negative numbers. This is called the commutative law of multiplication.

\[ 2 \times (-3) = (-3) \times 2 \]

\[ \boxed { \times } = \boxed { \times } \]

Calculate the following expressions \( \text{a} \) and \( \text{b} \), and compare the results.

\[ (2 \times (-3)) \times (-4) \]

\[ 2 \times ( -3 \times (-4)) \]

The following also holds for signed numbers. This is called the associative law of multiplication.

\[ (\boxed { \times } \times \boxed { \times }) \times \boxed { \times } = \boxed { \times } \times (\boxed { \times } \times \boxed { \times }) \]

Since the commutative and associative laws hold for multiplication, when multiplying signed numbers, you can rearrange the order or grouping without changing the result.

\[ (-15) \times 13 \times (-2) \]

\[ = (-15) \times (-2) \times 13 \]

\[ = (-15) \times (13) \times 2 \]

\[ = 30 \times 13 \]

\[ = 390 \]

Check 3 Calculate \((-4) \times (-9) \times 25\).

Problem 9 Calculate the following.

1. \((-3) \times 125 \times (-8)\)

2. \((-12) \times 45 \times \frac{1}{6}\)
If we multiply a nonzero number by a negative number, what happens to the sign of the original number? What if we multiply the number by two, three, or four negative numbers?

We can summarize the calculation of the product of several numbers as follows.

### Sign and absolute value of a product

The sign of a product is as follows:
- If there is an odd number of negative numbers, the sign is $-$. If there is an even number of negative numbers, the sign is $+$.

The absolute value of the product is the product of the absolute values of the individual numbers.

#### Example 6

1. $(-2) \times (-8) \times 2 \times (-4) \times (-5)$
   
   $= (2 \times 8 \times 2 \times 4 \times 5)$
   
   $= 640$

2. $(-\frac{1}{2}) \times (-8) \times (-\frac{7}{4})$
   
   $= -\left(\frac{1}{2} \times 8 \times \frac{7}{4}\right)$
   
   $= -7$

#### Note

If a plus or minus sign follows an operator (such as a multiplication sign), we use parentheses. For example, instead of writing “$3 \times -5$” we write “$3 \times (-5)$”.

#### Problem 10

Calculate the following.

1. $2 \times (-3) \times 9$
2. $(-6) \times (-3) \times 8$
3. $(-2) \times (-7) \times 5 \times (-4)$
4. $(-2) \times 2.5 \times (-1.4)$
5. $\left(-\frac{5}{3}\right) \times (-6) \times \left(-\frac{2}{3}\right)$
Powers

When we multiply two or more equal numbers, we can express this as follows:

- $5 \times 5$ is shown as $5^2$, is read as “five squared” or “five to the 2nd power”
- $2 \times 2 \times 2 = 2^3$, is read as “two cubed” or “two to the 3rd power”

When the same number is multiplied together in this way, the result is called a power. The superscript number to the right is called the exponent, indicating how many copies of the number were multiplied.

When the exponent is 2 or 3, the power is normally called a square or a cube.

**Example 7**

1. $(-2) \times (-2) \times (-2) = (-2)^3$
2. $\frac{3}{5} \times \frac{3}{5} = \left(\frac{3}{5}\right)^2$

**Check 4**

Express the following products as powers.

1. $7 \times 7$
2. $(-4) \times (-4) \times (-4)$
3. $\left(-\frac{1}{2}\right) \times \left(-\frac{1}{2}\right)$

**Example 8**

1. $(-3)^2 = (-3) \times (-3) = 9$
2. $-3^2 = -(3 \times 3) = -9$
3. $2 \times 3^2 = 2 \times 9 = 18$
4. $(2 \times 3)^2 = 6^2 = 36$

**Problem 11**

In the expressions in Example 8, identify the numbers that are squared.

**Check 5**

Calculate the following.

1. $(-1)^4$
2. $-5^2$
3. $(-3) \times 2^2$
4. $(2 \times 4)^3$
Division

What numbers fit in the boxes?

1. \( \square \times (+3) = +6 \)
2. \( \square \times (+3) = -6 \)
3. \( \square \times (-3) = +6 \)
4. \( \square \times (-3) = -6 \)

The calculation to find the numbers that fit in the boxes above is division.

The result of division is called the quotient of the numbers.

Division is the inverse operation of multiplication. We can make the following expressions for the divisions shown in examples 1 and 2 above.

1. Since \( ( +2 ) \times (+3) = +6 \) we have:
   \[ (+6) \div (+3) = +2 \]
2. Since \( ( -2 ) \times (+3) = -6 \) we have:
   \[ (-6) \div (+3) = -2 \]

Problem 1: Make division expressions for examples 3 and 4 above.

We can summarize the division of signed numbers as follows.

**Division of signed numbers**

To find the quotient of two signed numbers, proceed as follows.

1. If the numbers have the same sign, add a plus sign to the quotient of the absolute values.
2. If the numbers have different signs, add a minus sign to the quotient of the absolute values.

If zero is divided by a positive or negative number, the quotient is zero.

We do not consider division by zero.
Example 1
Quotients of numbers with the same sign
1. \((+12) \div (+4) = +(12 \div 4)\)
\(= +3\)
\(= 3\)
2. \((-12) \div (-4) = +(12 \div 4)\)
\((-) \div (-) \rightarrow (+)\)
\(= +3\)
\(= 3\)

Check 1
Calculate the following.
1. \((+18) \div (+3)\)
2. \((-32) \div (-4)\)

Example 2
Quotients of numbers with opposite signs
1. \((+12) \div (-4) = -(12 \div 4)\)
\((+) \div (-) \rightarrow (-)\)
\(= -3\)
2. \((-12) \div (+4) = -(12 \div 4)\)
\((-) \div (+) \rightarrow (-)\)
\(= -3\)

Check 2
Calculate the following.
1. \((+36) \div (-6)\)
2. \((-26) \div (+2)\)

Problem 2
Calculate the following.
1. \((-54) \div (-9)\)
2. \(0 \div (-7)\)
3. \(48 \div (-3)\)
4. \((-96) \div 8\)

Consider how to write fractions in which the numerator or denominator is a negative number.

Example 3
\(-\frac{3}{5} = (-3) \div 5 = -(3 \div 5) = -\frac{3}{5}\)
Therefore, \(-\frac{3}{5} = -\frac{3}{5}\)

Problem 3
Following Example 3, check that \(-\frac{3}{5}\)
Division and reciprocals

By what number should we multiply \(\frac{3}{4}\) to obtain 1 as the product?

If the product of two numbers is 1, we say that one number is the reciprocal of the other.

Example 4

1. The reciprocal of 3 is \(\frac{1}{3}\), and the reciprocal of \(\frac{1}{3}\) is 3.

\[3 \times \frac{1}{3} = 1\]

2. The reciprocal of \(-\frac{3}{4}\) is \(-\frac{4}{3}\).

\[\left(-\frac{3}{4}\right) \times \left(-\frac{4}{3}\right) = 1\]

Thus the reciprocal of a positive or negative number is the reciprocal of the absolute value, with the same sign attached. Since zero multiplied by any number is zero, it never produces the product 1, and has no reciprocal.

Check 3

Find the reciprocals of the following numbers.

1. \(\frac{3}{10}\)
2. \(-1\)
3. \(-\frac{15}{4}\)
4. 0.5

Tigger

Calculate the following expressions 4 and 5, and compare the results.

a. \(10 \div (-2)\)
b. \(10 \times \left(-\frac{1}{2}\right)\)

From what we have seen so far, we can say the following about division.

Division and reciprocals

Dividing by a positive or negative number is the same as multiplying by the reciprocal of the number.
Let's convert a division to a multiplication.

Example 5
\[
\frac{8}{9} \div \left(-\frac{2}{3}\right) = \frac{8}{9} \times \left(-\frac{3}{2}\right)
\]
\[= -\frac{4}{3}\]

Check 4
Calculate
\[
\frac{3}{2} \div \left(-\frac{5}{7}\right)
\]

Problem 4
Calculate the following.
1. \(\left(-\frac{2}{7}\right) \div (-4)\)
2. \(\left(-\frac{9}{5}\right) \div \left(-\frac{3}{4}\right)\)

Calculations with combined division and multiplication

Given an expression involving both multiplication and division, we can convert it to use multiplication only.

Example 6
Calculate
\[
8 \div \left(-\frac{14}{5}\right) \times (-7)
\]

Answer

Change the division to multiplication

Check 5
Calculate
\[
(-12) \div \frac{3}{4} \times (-8)
\]

Problem 5
Convert the following expressions to use only multiplication, and calculate.
1. \((-15) \times (-2) \div (-18)\)
2. \((-\frac{5}{3}) \times \frac{7}{15} \div \frac{5}{6}\)
Calculations Combining All Arithmetic Operations

We can refer to the operations of addition, subtraction, multiplication, and division together as the arithmetic operations. We will look at calculations involving all of these with signed numbers.

Example 1
Calculate
\[ 9 + 8 \times (-2) \]

Answer

In calculations involving all arithmetic operations do multiplication (or division) first. Then do addition (or subtraction).

Problem 1
Calculate the following.
1. \[ 2 \times (-5) + (-9) \]
2. \[ (-3) \times (-4) - (-5) \times 2 \]
3. \[ -5 - 9 \div (-3) \]
4. \[ 8 - (-4^2) \times (-2) \]

Example 2
Calculate
\[ 60 \div (-6 + 2) \]

Answer

In an expression with parentheses, calculate the part in parentheses first.

Problem 2
Calculate the following.
1. \[ 3 \times (-5 - 2) \]
2. \[ 36 \div (-13 + 4) \]
3. \[ (-2) \times (-4 + 8) - (-9) \]
4. \[ 8 + (4 - 3^2) \times 2 \]
Distributive law

In the following examples 1 and 2, calculate the values of the two expressions and compare the results.

1. \((-7) + (-3)\) \times 5, \((-7) \times 5 + (-3) \times 5\)
2. \((-2) \times \{(-23) + 3\}, (-2) \times (-23) + (-2) \times 3\)

The following relations also hold for signed numbers:

\[(\Box + \bigcirc) \times \bigtriangleup = \Box \times \bigtriangleup + \bigcirc \times \bigtriangleup\]

\[\bigtriangleup \times (\Box + \bigcirc) = \Box \times \bigtriangleup + \bigcirc \times \bigtriangleup\]

This is called the distributive law.

Problem 3 Calculate the following, using the distributive law.

1. \(\left(\frac{7}{9} - \frac{5}{6}\right) \times 18\)
2. \((-7) \times 47 + (-7) \times 3\)

Window on math ... Making 100

Using the numerals 1, 2, 3, 4, 5, 6, 7, 8, 9 in this sequence, insert +, -, x, and + signs to make an expression with a particular value. This puzzle is known in Japan as Komachi-zan, and dates back to the Edo period. It is not well known in the West, but the famous English puzzle writer H.E. Dudeney published the same puzzle in 1899, under the title “The Digital Century.”

Insert a +, -, x, or + sign into each box in the following expressions, so that each correctly evaluates to 100.

1. \(12 - 3 + 4 + 5 - 6 + 7 + 89 = 100\)
2. \(1 + 2 - 34 - 56 + 78 + 9 = 100\)
3. \(12 + 3 - 4 + 5 + 6 + 7 + 8 + 9 = 100\)
4. \(1 + 2 \times 3 - 4 \times 5 + 6 + 7 + 8 \times 9 = 100\)
5. \(-123 - 4 + 5 + 6 + 7 + 8 + 9 = 100\)
Using Signed Numbers

We'll look at everyday problems using signed numbers.

The map on the right shows time differences in cities around the world, taking Tokyo as the reference. The time differences are shown by signed numbers, as in the map. From the map, you can see that to get Wellington time, we add 3 hours to Tokyo time, and to get London time, we subtract 9 hours from Tokyo time.

If, for example, the time in Tokyo is 21:00, the time in London is nine hours earlier, that is 12:00.

Problem 1 When the time in Tokyo is 15:00, find the time in the following cities.

1. Wellington
2. London
3. New York

Example 1 Find the time difference between London and Sydney.
Taking Tokyo as reference:

London time is \(-9\) hours
Sydney time is \(+1\) hour

Therefore the time difference between London and Sydney, taking London as the reference is:

\[(+1) - (-9) = +10\]

Thus, +10 hours.

Problem 2 What is the time difference between London and Beijing, taking London as the reference?
The capacity of an elevator is calculated on the basis of an average person weighing 65 kg. Will an elevator with a rated capacity of six persons be able to carry people with the following weights:

- 62 kg, 57 kg, 70 kg, 68 kg, 75 kg, 60 kg

**Answer**

Express the weights as differences from 65 kg, thus:

-3 kg, -8 kg, +5 kg, +3 kg, +10 kg, -5 kg

The sum of these numbers is:

\[ (-3) + (-8) + 5 + 3 + 10 + (-5) = 2 \]

So the weight is 2 kg over the limit.

Answer: No, the elevator cannot carry these people.

**Problem 3**

I bought five cakes for 198 yen each, and three cakes for 208 yen each. What was the total cost? Use the method of Example 2 to calculate the answer.

---

**Window on math ... Simple average calculation**

The average value of 10, 15, and 17 is 14. We can calculate this as follows.

1. Subtract 15 from each number, giving -5, 0, and 2, then find the average of these values.
   \[ (-5 + 0 + 2) ÷ 3 = -1 \]

2. Add the value found in step 1 to the 15 that was originally subtracted.
   \[ (-1) + 15 = 14 \]

In the calculation above, 15 is called a “provisional average.” By choosing a value which looks to be close to the actual average, we can often simplify the calculation.

For the following numbers, think of a provisional average, and use it to find the actual average.

23, 24, 29, 27, 22

---

1 — *Positive and Negative Numbers*
Basic Exercises

**Multiplication**

Calculate the following.

1. \((-7) \times 6\)
2. \((-4) \times (-8)\)
3. \(4 \times (-12)\)
4. \((-3) \times (-16)\)
5. \(\frac{2}{3} \times \left(-\frac{3}{7}\right)\)
6. \((-5) \times 7 \times (-2)\)

**Powers**

Calculate the following.

1. \((-7)^2\)
2. \(-7^2\)
3. \(3 \times (-2)^3\)

**Division**

Calculate the following.

1. \(15 \div (-3)\)
2. \((-56) \div 7\)
3. \(48 \div (-8)\)
4. \((-38) \div (-2)\)
5. \((-52) \div 4\)
6. \(\left(-\frac{2}{5}\right) \div \left(-\frac{3}{5}\right)\)

**Calculations combining multiplication and division**

Calculate the following.

1. \((-4) \times 8 \div (-2)\)
2. \((-6) \div \frac{3}{5} \times 10\)

**Calculations combining all of the arithmetic operations**

Calculate the following.

1. \(5 - 2 \times (-3)\)
2. \((-5) \times (-2 + 4)\)
Chapter Summary Problems • A

1. For each of the following sets of numbers, use inequality signs to show a comparison of the values.
   1) 5, -9
   2) -4, -1
   3) 3, -5, -2

2. Write the integers with an absolute value less than 5, in sequence from lowest to highest value.

3. Find the numbers that fit in the boxes in the following equations.
   1) (-8) + | ] = -8
   2) (-8) + | ] = 0
   3) (-8) \times [ ] = -8
   4) (-8) \times [ ] = 1

4. Calculate the following.
   1) -7 + (-10)
   2) -3 - (-9)
   3) -6 + 14
   4) 12 - 18
   5) -1.5 - (-1.1)
   6) -\frac{4}{9} + (-\frac{2}{9})
   7) 3 - 7 - 2 + 5
   8) 5 - 3 + 4 - 11 - 2
   9) 8 - 12 - (-5)
   10) -1 - (-9) + (-8) - 6

5. Calculate the following.
   1) (-7) \times (-8)
   2) (-45) \div 15
   3) 8 \times (\frac{-5}{12})
   4) (-4) \div (-\frac{2}{3})
   5) (-4)^2 \times (-5)
   6) (-20) \div (-4) \times (-5)
   7) -15 + (-10) \div (-2)
   8) (-6) \times (-3 + 10)

6. From the numbers -2, -1, 0, 1, and 2, find all numbers that fit each of the following statements.
   1) Subtracting 1 results in a negative number.
   2) Multiplying by -2 then adding 3 results in a positive number.
Chapter Summary Problems • B

1. Calculate the following.
   ① \((-3) \times \left(10 \div (6 - 8)\right)\)
   ② \(4 - (-3)^2 \times (-2) + (-2)\)
   ③ \(\frac{1}{4} \times \left(-\frac{5}{6}\right) \div \left(-\frac{5}{8}\right)\)
   ④ \(\frac{1}{3} - \left(-\frac{1}{2}\right)^2 \div \left(-\frac{3}{8}\right)\)

2. In the following table, the values in row A show the number of customers in a store for each day from Monday to Friday. The values in row B show the differences of the values in row A from a reference number of customers, using a positive value if more than the reference value, and a negative value if less.

<table>
<thead>
<tr>
<th>Monday</th>
<th>Tuesday</th>
<th>Wednesday</th>
<th>Thursday</th>
<th>Friday</th>
</tr>
</thead>
<tbody>
<tr>
<td>A 123</td>
<td>153</td>
<td>112</td>
<td>150</td>
<td></td>
</tr>
<tr>
<td>B -7</td>
<td>+23</td>
<td></td>
<td></td>
<td>-23</td>
</tr>
</tbody>
</table>

   ① Write the missing numbers in the table.
   ② Find the average number of customers per day over the five days.

Let's investigate!

Look for examples in use in everyday life where positive and negative values are used to show something.

On the golf course (Mie Prefecture)
In the following diagram, we are using matchsticks to form squares. How many matchsticks does it take to make 20 squares?
To make five squares, how many matchsticks do we need?
Try drawing a diagram.

Alice found the number of matchsticks needed to make five squares using the following method:

First we need four matchsticks to make one square, then three matchsticks for each square after that, so the total is:

Bob's way is different than Alice's, and he thought of the following expression to represent the number of matchsticks needed:

\[1 + 3 \times 5\]

How do you think he thought about the matchsticks to obtain this answer? Try to see how, using the following diagram. Can you think of other expressions for the answer?
On the previous page, Bob thought of the following diagram as a way of finding the number of matchsticks needed for one, two, and three squares.

<table>
<thead>
<tr>
<th>Number of squares</th>
<th>Expression for the number of matchsticks</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1 + 3 \times \text{(number of squares)}</td>
</tr>
<tr>
<td>2</td>
<td>1 + 3 \times \text{2}</td>
</tr>
<tr>
<td>3</td>
<td>1 + 3 \times \text{3}</td>
</tr>
</tbody>
</table>

We can see from the diagram above that the number of matchsticks is always given using the following expression:

\[ 1 + 3 \times \text{(number of squares)} \]

The number of squares can be 1, 2, 3, ... or any other number, but if we write the letter \( x \) to represent this number, then the total number of matchsticks can be represented by:

\[ (1 + 3 \times x) \]

The technique of writing a letter to stand for a quantity in this way is called algebra.

**Problem 1**

To make 20 squares with matchsticks in this way, how many matchsticks are needed? How many are needed for 50 squares?

The number of matchsticks needed depends only on the number of squares being made. When we use the letter \( x \) to write the expression \( 1 + 3 \times x \), this covers all cases.

We can think of the expression \( 1 + 3 \times x \) both as a way of finding the number of matchsticks, and also as a way to express the result of the calculation.

**Problem 2**

Use the letter \( x \) to write an expression to represent the way Alice thought of calculating on the previous page, and for any other ways that you may have thought of.
Let's try using letters to represent various quantities.

Example 1
If I buy \( x \) notebooks, each costing 90 yen, the total cost is:

\[
(90 \times x)
\]

---

Check 1
I bought \( x \) meters of rope costing 60 yen per meter. Write an expression for the total cost, using the letter \( x \):

---

Problem 3
Answer the following four questions, with expressions using letters.

1. In a class of \( n \) students, 3 were absent. How many were present?
2. What is the perimeter of a square when one side is \( a \) cm?
3. I divide \( x \) meters of ribbon equally among four people. How many meters of ribbon does each person receive?
4. The air temperature at 9 o'clock is \( t \) °C, and the temperature at 10 o'clock is 3 degrees higher. What is the air temperature at 10 o'clock?

In example 1, \( x \) stood for a natural number, 1, 2, 3, ... but it can also be used to stand for a fraction or negative number such as 0.5 or -4.

Problem 4
In Problem 3, which questions do the letters stand for numbers that may be fractions?
In which questions do the letters stand for numbers that may be negative?
Writing Algebraic Expressions

We will learn about expressing various quantities using algebra, and about conventions used when writing these algebraic expressions.

Representing Expressions with Multiplication

In a multiplication expression including a letter, such as $3 \times x$, we omit the multiplication sign, and write $3x$. Thus we can rewrite the expression $1 + 3 \times x$ on page 44 as $1 + 3x$.

In a product of a letter and a numeral, we write the numeral first.

Example 1
I paid for two items each costing $a$ yen with a 1000 yen bill.
In this case the change can be presented as: 

\[(1000 - 2a)\text{ yen.}\]

And we can write this omitting the multiplication sign as:

\[(1000 - 2a)\text{ yen.}\]

Check 1
I bought $x$ apples costing 120 yen each. Write the total cost as an algebraic expression.

Problem 1
Write algebraic expressions for the following quantities:

1. The length of tape remaining if two strips, $a$ cm long, were cut from a piece of tape 80 cm in length.
2. The area of a rectangle with a height of 5 cm and a width of $y$ cm.
3. The total cost of six pencils costing $x$ yen each and one eraser costing 100 yen.
I bought some stamps: $x$ at 50 yen, and $y$ at 80 yen. The total cost was:

\[(50 \times x + 80 \times y) \text{ yen.}\]

By omitting the multiplication sign we can write this as:

\[(50x + 80y) \text{ yen.}\]

**Problem 2** Write algebraic expressions for the following quantities.

1. The total weight of three items weighing $a$ grams each and two items weighing $b$ grams each.
2. The amount remaining in a kitty if five people each put $a$ yen into the kitty, which is used to buy $b$ apples at 120 yen each.

We can summarize the way we write products in algebra as follows:

**Writing expressions using multiplication**

1. When writing expressions using letters, omit the multiplication sign.
2. When writing expressions using numbers and letters, write the number first.

\[\begin{align*}
4 \times x &= 4x \\
y \times 7 &= 7y
\end{align*}\]

**Note**

The expression $b \times a$ can be written as $ba$, but when writing expressions using letters, we often rearrange the letters in alphabetical order, such as $ab$.

**Check 2** Rewrite the following expressions.

1. $x \times y$
2. $c \times a \times b$
3. $a \times x \times 2$
4. $(a - b) \times 5$

**Problem 3** Rewrite the following as algebraic expressions.

1. Two times the product of $x$ times $y$
2. Six times the sum of $x$ plus $y$
We write letters multiplied by 1 or multiplied by a negative number as follows:

For $1 \times a$ we write $a$.

For $(-5) \times a$ we write $-5a$.

For $(-1) \times a$ we write $-a$.

Check 3 Rewrite the following expressions:

1 $(-4) \times x$
2 $y \times 1$
3 $b \times (-1)$

Problem 4 Rewrite the following expressions:

1 $a \times (-2) + 1$
2 $x \times (-1) + y \times 1$
3 $b - 0.1 \times a$

What can we omit?

Trigger How would we write an expression to represent the volume of a cube with sides $a$ cm long?

We write a number multiplied by itself as follows:

- **Writing powers**

  Write a number multiplied by itself using power notation.

  The base of a power is the factor and the exponent of a power is the number of times the factor is used.

  Example 4

  1 $a \times a \times a = a^3$
  2 $x \times 4 \times x = 4x^2$
  3 $x \times x \times y \times y \times y = x^2y^3$

  Check 4 Rewrite the following expressions:

  1 $x \times x \times 3$
  2 $x \times y \times y$
  3 $a \times b \times a \times a \times b$
Representing Expressions with Division

If I buy five of the same notebook and the cost is $a$ yen, how would we write an algebraic expression for cost of each notebook?

We write expressions involving division and letters as follows:

Writing expressions using division

For expressions involving division and a letter, instead of using the division sign ($\div$), we write a fraction.

\[ a \div 5 = \frac{a}{5} \]

Example 5

1. \[ a \div 9 = \frac{a}{9} \]
2. \[ 3x \div 4 = \frac{3x}{4} \]
3. \[ (x + 3) \div 2 = \frac{x + 3}{2} \]
4. \[ x \div (-2) = \frac{x}{-2} = -\frac{x}{2} \]

Note

Since \( a \div 9 = a \times \), we can also write as \( a \). Similarly, we can write as \( a \).

Check 5

Rewrite the following expressions.

1. \( x \div 7 \)
2. \( 5x \div 2 \)
3. \( (a + b) \div 6 \)
4. \( a \div (-3) \)

Problem 5

Answer the following questions.

1. If we divide \( a \) liters of orange juice equally among three people, how many liters does each person receive?

2. How many minutes does it take to travel \( x \) meters at a speed of 70 meters per minute?
Problem 6 Write out the following expressions, using multiplication and division signs (×, ÷).

1. $6ab$
2. $5x^2$
3. $\frac{2x}{7}$
4. $\frac{a-b}{2}$

Let's look at what quantities algebraic expressions represent.

Example 6 Suppose we ride in a car traveling at a speed of $a$ km per hour for $b$ hours. Then $ab$ represents the distance traveled, in kilometers.

Note The speed of a km per hour is often written as $a$ km/h.

Problem 7 A rectangle is $a$ cm high and $b$ cm wide. In this case, what do the following expressions represent? Give the units.

1. $ab$
2. $2(a+b)$

Window on math ... Edo period notation

Mathematical texts written in the Edo period use Chinese characters like 甲 and 乙 to represent expressions as follows.

<table>
<thead>
<tr>
<th>Modern notation</th>
<th>简单符号</th>
<th>繁体符号</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a+b$</td>
<td>甲乙</td>
<td>甲乙</td>
</tr>
<tr>
<td>$a-b$</td>
<td>甲乙</td>
<td>甲乙</td>
</tr>
<tr>
<td>$a \times b$</td>
<td>甲乙</td>
<td>甲乙</td>
</tr>
<tr>
<td>$a \div b$</td>
<td>甲乙</td>
<td>甲乙</td>
</tr>
</tbody>
</table>
Substituting and Evaluating a Variable Expression

When we were calculating the number of matchsticks on page 42, the number required to make x squares was \((1 + 3x)\).

To make 60 squares, how many matchsticks are required?

In the expression, we replace the x in \(1 + 3x\) with 60 to carry out the calculation.

\[
\begin{align*}
1 + 3x & \\
1 + 3 \times 60 & \\
\end{align*}
\]

Problem 1 In the expression, to make 36 squares, how many matchsticks are needed?

When we replace a letter in the expression with a number substituting the value of the letter, and the result of the calculation after substitution is called the evaluation of the expression.

Example 1 When \(x = 3\), find the value of \(5 - 4x\). Also find the value when \(x = -2\).

Answer

When \(x = 3\)

\[
\begin{align*}
5 - 4x & \\
& = 5 - 4 \times 3 \\
& = 5 - 12 \\
& = -7 \\
\end{align*}
\]

Answer: -7

When \(x = -2\)

\[
\begin{align*}
5 - 4x & \\
& = 5 - 4 \times (-2) \\
& = 5 + 8 \\
& = 13 \\
\end{align*}
\]

Answer: 13

When substituting a negative value, put it in parentheses.

Check 1 Find the values of the following expressions when \(x = 2\). Also find the values when \(x = -4\).

1. \(5x + 2\)
2. \(3 - 2x\)

Problem 2 Find the values of the following expressions when \(x = \frac{1}{2}\).

1. \(4x + 5\)
2. \(1 - 3x\)
When $a = -2$, find the values of $-a$ and $a^2$.

$$-a = -( -2)$$
$$= 2$$

$$a^2 = (-2)^2$$
$$= (-2) \times (-2)$$
$$= 4$$

Check 2 Find the values of the following expressions when $a = -5$.

1. $-a$
2. $a^2$

Problem 3 Find the values of the following expressions when $a = -\frac{1}{3}$.

1. $-a^2$
2. $(-a)^2$

Example 3 The speed of sound through air depends on the air temperature. When the air temperature is $t \, ^{\circ}C$, the speed can be shown as:

$$(331.5 + 0.6t) \, \text{m/s (meters per second)}$$

Find the speed of sound when the air temperature is 20 °C.

Answer Substituting 20 for $t$ in $331.5 + 0.6t$ gives us:

$$331.5 + 0.6 \times 20 = 331.5 + 12$$
$$= 343.5$$

Answer 343.5 m/s

Problem 4 Find the speed of sound when the air temperature is 0 °C. Also find the speed at -10 °C.

Problem 5 In an air temperature of 30 °C the sound of thunder was heard 2 seconds after the flash of lightning. How far away is the lightning?
Basic Exercises

1. **Conventions for writing algebraic expressions**
   
   Rewrite the following expressions:
   
   1) \( b \times 9 \)
   2) \( b \times 4 \times a \)
   3) \( y \times y \)
   4) \( (x+3) \times 2 \)
   5) \( 3x \div 2 \)
   6) \( (a-2) \div 3 \)

2. Rewrite the following expressions:
   
   1) \( -1 \times t \)
   2) \( c \times 1 \)
   3) \( a \times (-2) \)

3. Rewrite the following expressions using multiplication and division signs (\( \times, \div \)).
   
   1) \( -6a \)
   2) \( 2x^2 \)
   3) \( \frac{x}{4} \)

4. **Representing quantities**
   
   Write expressions for the following quantities.
   
   1) The total cost for \( x \) notebooks at 90 yen each and \( y \) pencils at 50 yen each.
   2) The weight of 1 meter of wire if 10 meters weighs \( a \) grams.
   3) The sum of \( y \) and twice \( x \).
   4) Twice the sum of \( x \) and \( y \).

5. **Values of expressions**
   
   Find the values of the following expressions when \( a = 3 \).
   
   1) \( 2a \)
   2) \(-3a + 5\)
   3) \( a^2 \)

6. Find the values of the following expressions when \( a = -2 \).
   
   1) \( 7a \)
   2) \( 4a - 3 \)
   3) \( 5a^2 \)
Calculating with Algebraic Expressions

1 Linear Calculation

Terms and Coefficients

In the expression $1 + 3x$, the parts separated by the addition sign are:

1 and $3x$

These are called **terms**.

In term $3x$, the number 3 is called the **coefficient** of $x$.

A term may include letters, as in the case of $3x$, or it may consist of just a number such as 1.

**Example 1**

Let's find the terms and coefficients in $x - 3y$.

$$x - 3y = x + (-3y)$$

So the terms are $x$ and $-3y$.

The coefficient of $x$ is 1

The coefficient of $y$ is $-3$

Check 1 Find the terms and coefficients of $2x + 3y$.

Problem 1 Find the terms and coefficients of the following expressions.

1. $4x + y$
2. $-x + \frac{2}{3}y$
3. $5a - \frac{b}{4}$

A term such as $3x$ in the expression $1 + 3x$, including just one letter, is called a linear term. An expression which can be written as the addition of linear terms and numbers is called a linear expression.
What expression would represent the sum of the areas of the two rectangles shown on the right?

When we use letters in algebra to represent quantities, common letters must represent the same value. Therefore, we can collect terms with common letters together into a single term, making the expression simpler.

Example 2

1. \(3x + 6x = (3 + 6)x\)
   \[= 9x\]

2. \(3x - 6x = (3 - 6)x\)
   \[= -3x\]

Check 2

Calculate the following:

1. \(5x + 7x\)
2. \(6y - 2y\)

Problem 2

Calculate the following:

1. \(4x - 3x\)
2. \(-9a + 6a\)
3. \(-3y - y\)
4. \(7a + (-8a)\)

Example 3

\(7x + 3 - 5x - 6\)
\[= 7x - 5x + 3 - 6\]
\[= (7 - 5)x + 3 - 6\]
\[= 2x - 3\]

Check 3

Calculate \(8x + 6 - 3x - 3\).

Problem 3

Calculate the following:

1. \(5x - 8 + 3x - 2\)
2. \(6a - 4 - 5a + 1\)
3. \(3x - 2 + x + 9\)
4. \(a + 5 - 7a - 4\)

Terms with common letters are called similar terms.
Adding and Subtracting Linear Expressions

Two sisters, Alice and Belinda went shopping. Alice bought five pencils at \( x \) yen each, and one eraser at 60 yen. Belinda bought three of the same pencils as Alice, and one eraser at 50 yen.

Try writing expressions for each of the following in this example.

1. How much did each sister spend?
2. What was the total spent by the two sisters?
3. How much more did Alice spend than Belinda?

To add linear expressions, put terms with common letters together, and numeric terms together.

\[
(3a + 2) + (6a - 5) = 3a + 2 + 6a - 5 = 3a + 6a + 2 - 5 = 9a - 3
\]

You can add linear expressions using the method on the right:

\[
\begin{align*}
3a + 2 + 6a - 5 &= 9a - 3
\end{align*}
\]

Check 4 Calculate the following:

1. \((5a + 4) + (2a - 1)\)
2. \((3x - 2) + (2x + 5)\)

Problem 4 Calculate the following:

1. \((4a - 3) + (5a + 6)\)
2. \((2x + 5) + (3x - 7)\)
3. \((5x - 9) + (x - 9)\)
4. \((-2a + 7) + (3a - 6)\)
5. \((4a + 1) + (-5a + 3)\)
6. \((7x - 5) + (-9x + 5)\)
7. \((-2 + a) + (-3 - a)\)
8. \((a - 2) + (6 - 9a)\)
Subtraction is performed the same as addition. To subtract linear expressions, reverse the addition and subtraction signs of the expression to be subtracted, and then add.

**Example 5**

\[(a + 7) - (5a - 3)\]

\[= (a + 7) + (-5a + 3)\]

\[= a + 7 - 5a + 3\]

\[= -4a + 10\]

**Check 5**

Calculate \((9a + 7) - (2a + 3)\)

**Problem 5**

Calculate the following:

1. \((6x - 3) - (4x + 5)\)
2. \((a - 10) - (2a - 1)\)
3. \((-2x - 9) - (x + 2)\)
4. \((5a - 4) - (4 - 3a)\)

---

**Multiplying Linear Expressions by Numbers**

**Example 6**

1. \(5a \times 3\)

\[= 5 \times a \times 3\]

\[= 5 \times 3 \times a\]

\[= 15a\]

2. \(2x \times \frac{1}{2}\)

\[= 2 \times x \times \frac{1}{2}\]

\[= 2 \times \frac{1}{2} \times x\]

\[= x\]

3. \((-x) \times 3\)

\[= (-1) \times x \times 3\]

\[= -3x\]

**Check 6**

Calculate the following:

1. \(4a \times 4\)
2. \(6x \times \frac{1}{3}\)
3. \((-a) \times 5\)

**Problem 6**

Calculate the following:

1. \(3n \times 8\)
2. \((-2) \times 4a\)
3. \(\frac{1}{5} y \times 5\)
4. \((-x) \times (-6)\)
A rectangle has a height of 3 cm and a width of \( a \) cm. What expression represents the length of the perimeter of the rectangle?

To multiply a linear expression by a number, we can use the distributive law:

\[
ab + ac 
\]

**Example 7**

\[
2(a + 3) = 2 \times a + 2 \times 3 = 2a + 6
\]

**Example 8**

\[
(3x - 5) \times (-2) = 3x \times (-2) + (-5) \times (-2)
\]

\[
= -6x + 10
\]

\[
(3x - 5) \times (-2) = -6x + 10
\]

\[
(2) \quad -(4a + 5) = (-1) \times (4a + 5)
\]

\[
= -4a - 5
\]

**Check 7** Calculate the following:

1. \( 3(a + 2) \)
2. \( (2x - 1) \times (-6) \)

**Problem 7** Calculate the following:

1. \( 5(2b - 5) \)
2. \( -4(c + 5) \)
3. \( -(6a - 9) \)
4. \( 6(\frac{3}{2}x + 2) \)
5. \( \frac{1}{2}(4x + 8) \)
6. \( (\frac{2}{3}x - \frac{1}{4}) \times 24 \)
Check 8 Calculate

Problem 8 Calculate the following:

1 \[ \frac{3x + 5}{4} \times 12 \]
2 \[ -15 \times \frac{5x - 3}{3} \]

Dividing Linear Expressions by Numbers

Example 10

1 Divide \( 12a + 6 \) by 3.
\[
\frac{12a + 6}{3} = (12a + 6) \times \frac{1}{3} = 12a \times \frac{1}{3} + 6 \times \frac{1}{3} = 4a + 2
\]

2 Divide \( 12a + 6 \) by \(-3\).
\[
\frac{12a + 6}{-3} = (12a + 6) \times \left(-\frac{1}{3}\right) = 12a \times \left(-\frac{1}{3}\right) + 6 \times \left(-\frac{1}{3}\right) = -4a - 2
\]

As shown in Example 10, we can carry out division of a linear expression by a number, by first converting it to multiplication.

Check 9 Calculate the following:

1 \((16x - 8) \div 4\)
2 \((18x - 14) \div (-2)\)

Problem 9 Calculate the following:

1 \((20x + 15) \div 5\)
2 \((16a - 8) \div (-8)\)
3 \((48x - 24) \div (-4)\)
4 \((180x + 120) \div 60\)
More Calculations

Example 11
Calculate $2(x + 3) - 3(2x - 1)$.

Hint: Use the distributive law to eliminate parentheses, and collect terms with common letters.

Answer:

Problem 10 Calculate the following:

1. $7x + 2(3x - 5)$
2. $6(x + 2) + 5(2x - 3)$
3. $5(x - 2) + (-2x + 4)$
4. $2(a + 1) - 3(a - 4)$
5. $2(a - 1) - 3(a + 4)$
6. $5(4x - 2) - 4(5x - 2)$

Problem 11 Calculate $\frac{1}{2}(4x - 6) + \frac{2}{3}(3x + 9)$.

Problem 12 In the Trigger question on page 42, Alice and Bob each found expressions to represent the number of matchsticks needed to make $x$ squares, as follows:

Alice: $4 + 3(x - 1)$
Bob: $1 + 3x$

Substitute numerical values for $x$ in Alice's expression or one you found yourself in Problem 2 on page 44, calculate the expressions, and compare the results.

* Using the distributive law to eliminate the parentheses is called “multiplying out”.

60 1 — Algebra
Multiplication

The area of a parallelogram is represented as:

\[ \text{Area} = \text{Base} \times \text{Height} \]

If the base is \( a \) cm long, the height is \( h \) cm, and the area is \( S \) cm\(^2\), the following algebraic formula can be used.

\[ S = ah \]

**Problem 1**

If the length of the base of a triangle is \( a \) cm and the height is \( b \) cm, write a formula for the area \( S \) cm\(^2\).

In calculating the circumference or area of a circle, we use the value \( \pi \) (pronounced the same as "pi"), which is the ratio of the circumference of a circle to its diameter. Its value is approximately 3.1415926535897932384...

\[ \pi = \frac{\text{Circumference}}{\text{Diameter}} \]

**Example 1**

Find the formula for the circumference of a circle whose radius is \( r \) cm.

The diameter of the circle is \( 2r \) cm, therefore the circumference \( \ell \) cm is represented as:

\[ \ell = 2r \times \pi \]

Thus:

\[ \ell = 2\pi r \]

Since \( \pi \) is a single letter representing a fixed number, in a multiplication expression we usually write it after the number and before the other letters.

**Problem 2**

Write a formula for the area \( S \) cm\(^2\) of a circle of radius \( r \) cm.

**Problem 3**

Write the circumference and area of a circle with a radius of 4 cm, using \( \pi \).
Problem 4 Two circles have the same center at point O, and radii of 3 cm and 5 cm. Find the area of the ring-shaped area between the two circles, using \( \pi \).

### Calculating linear expressions

1. Calculate the following:

   1. \( 5x + 2x \)  
   2. \( 7x - x \)
   3. \( 2a - 4a + 5a \)
   4. \( x - 6 - 2x + 8 \)
   5. \( (6x + 5) + (4x - 3) \)
   6. \( (4a - 3) - (3a - 1) \)

2. Calculate the following:

   1. \( 2x \times (-4) \)  
   2. \( 30a \times \frac{1}{5} \)
   3. \( 5(3x + 2) \)
   4. \( (3a - 2) \times (-3) \)
   5. \( (10x - 25) \div 5 \)
   6. \( (15x + 9) \div (-3) \)

3. Calculate the following:

   1. \( 6(2x - 1) + (9x + 3) \)  
   2. \( 3(2x + 7) - 4(3x + 2) \)

### Multiplication

4. A rectangular parallelepiped has length of \( a \) cm, a width of \( b \) cm, and a height of \( c \) cm. Write a formula for its volume \( V \) cm\(^3\).
Chapter Summary Problems • A

1. Find the values of the following expressions when \( x = 3 \). Then find the values when \( x = -3 \).

1. \( 2x - 5 \)
2. \( 2x^2 - x \)

2. Calculate the following:

1. \( -3x + 8x \)
2. \( 6y - 7y \)
3. \( x - (-2x) \)
4. \( \frac{3}{5} x - x \)
5. \( -3x + 4x - x \)
6. \( \frac{4}{9} x - \frac{5}{9} x + \frac{7}{9} x \)
7. \( (a - 4) + (3a + 5) \)
8. \( (-2a - 4) - (4 - 2a) \)

3. Calculate the following:

1. \(-6(2x - 7)\)
2. \((6a - 18) \times \frac{5}{6}\)
3. \(\left(\frac{5}{6}x - \frac{3}{4}\right)x 12\)
4. \((-27x + 36) \div (-9)\)
5. \(3(x - 6) + 2(2x - 9)\)
6. \(2(3a - 7) + 5(-a + 3)\)
7. \(3(2a - 4) - 4(a - 5)\)
8. \(2(x - 7) - 4( -x + 2)\)

4. In each of the following examples, (1) and (2), find the sum of the two expressions. Also find the difference, subtracting the right-hand expression from the left-hand expression.

1. \(5x, 3x + 8\)
2. \(-3y - 1, 2y + 1\)

5. Answer the following questions:

1. Find the area of a rectangle whose height is 4 cm, and whose width is \( a \) cm more than its height.
2. I tried to give candy to \( x \) children, giving each one \( y \) pieces, but I was 17 pieces short. How many pieces of candy did I have altogether?
3. What is the cost of buying \( b \) grams of meat priced at \( a \) yen per 100 grams.
Chapter Summary Problems  •  B

1. Calculate the following:
   \[ \left( \frac{2}{3} x + \frac{4}{5} \right) - \left( -\frac{1}{2} x - \frac{1}{5} \right) \quad \text{and} \quad x - 4 - \frac{1}{3} (2x - 6) \]

2. Calculate the values of the following expressions when \( A = 2x - 5 \) and \( B = -x + 3 \).
   \[ 2A + B \quad \text{and} \quad 3A - 2B \]

3. Answer the following questions:
   ① How much is \( \frac{a}{10} \) of 2000 yen?
   ② A school had \( x \) students last year, and this year the number of students has increased by \( p\% \). How many students are there this year?

4. If the price of a pencil is \( a \) yen, and a notebook is \( b \) yen, what quantity does the expression \( 1000 - (3a + b) \) represent?

5. "Go" stones are arranged to make squares as shown below. How many "Go" stones are needed to make \( x \) squares?

   Four stones on each side

   Make \( x \) squares

   \[ \text{Four stones on each side} \]

   \[ \text{Make } x \text{ squares} \]

**Let's investigate!**

Try using matchsticks to make all sorts of different shapes.
Try to find expressions for the number of matchsticks needed to make \( x \) of these shapes joined together.
Cryptarithmetic

In the following calculations, each different character or letter represents a particular digit. Can you find the digits that fit?

+ )

× )

KYOTO

+ ) OSAKA

TOKYO
Let's look at various problems using a balance.

We have three weights, one each of 1 g, 3 g, and 9 g. Using these three weights and a balance, can we weigh any amount from 1 g to 13 g in 1 g increments? Try to work this out by increments using the figures on the opposite page.

Place two steel balls, a 1 g weight and a 9 g weight on the balance as shown in the diagram on the right.

If each ball weighs 1 g, will the scale balance?
What if each ball weighs 2 g?
1 Equations

Equalities

Look at the picture of the balance on the right. When the weight of a ball is \( x \) g, the sides balance. We can show the relation between the weights using an equals sign (\( = \)) as follows:

\[
2x + 1 = 9 \quad \text{...................... (1)}
\]

When we use an equals sign to indicate the relation between two quantities that are the same, it is called an equality. We call the part to the left of the equals sign the left hand side (LHS), and the part to the right the right hand side (RHS).

\[
2x + 1 = 9
\]

We'll show relations between quantities using equalities.

Example 1: Two pieces of string \( x \) cm long were cut from a length of string \( a \) cm long, leaving 5 cm. If we think about the length of string remaining, we can write the following:

\[
(\text{Original length}) - (\text{Length of two pieces cut off}) = (\text{Remaining length})
\]

This gives us the following relation between the quantities:

\[
a - 2x = 5
\]

Problem 1: In Example 1, we can also write an equation as follows:

\[
x = a
\]

Write an expression that fits on the left hand side.

Check 1: Write an equality representing each relation among the following quantities:

1. Three notebooks at \( a \) yen each cost the same as five pencils at \( b \) yen each.
2. The total weight of \( x \) items weighing 30 g each and one item weighing 60 g is 180 g.

3 — Equations
Looking at the equality (1) on the previous page, $2x + 1 = 9$, find the value of the left hand side when the value of $x$ is 1, 2, ..., 6.

<table>
<thead>
<tr>
<th>$x$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>LHS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When the value of $x$ is 4, the values of the left and right hand sides of the equality $2x + 1 = 9$ are the same, and thus the equality remains true.

An equality which remains true when certain values are substituted for the letters, and does not remain true for others is called an equation. A substituted value for which the equation remains true is called a solution of the equation. For example, the solution of the equation $2x + 1 = 9$ is 4.

Finding the solutions of an equation is called solving the equation.

---

**Example 2**

Which value, 0, 1, 2, or 3, is the solution to the equation $3x - 1 = 8$?

**Hint**

To check whether a number is the solution, just substitute the number for $x$, and see if the equality remains true.

**Answer**

When $x = 0$: $3x - 1 = 3 \cdot 0 - 1 = -1$

When $x = 1$: $3x - 1 = 3 \cdot 1 - 1 = 2$

When $x = 2$: $3x - 1 = 3 \cdot 2 - 1 = 5$

When $x = 3$: $3x - 1 = 3 \cdot 3 - 1 = 8$

Therefore the equality holds when $x = 3$.

**Check 2**

Which value, $-1$, 0, or 1, is the solution to the equation $2x + 5 = 7$?

**Problem 2**

Which value, 0, 1, 2, or 3, is the solution to the equation $4x - 1 = x + 5$?
Properties of Equalities

To solve an equation we have to convert the original equation to the form \( x = \square \). Let's look at the first equation on page 68, \( 2x + 1 = 9 \).

\[
2x + 1 = 9 \\
\downarrow \quad \downarrow \quad \text{Subtract 1 from each side} \\
2x = 8 \\
\downarrow \quad \downarrow \\
x = 4
\]

Problem 3 What do we have to do above to convert \( 2x = 9 \) into \( x = 4 \)?

To convert an equation this way, we use the following properties of equalities.

<table>
<thead>
<tr>
<th>Properties of equalities</th>
</tr>
</thead>
</table>
| 1. Adding the same number or expression to both sides of an equality leaves the equality remaining true. 
  If \( A = B \), then \( A + C = B + C \) |
| 2. Subtracting the same number or expression from both sides of an equality leaves the equality remaining true. 
  If \( A = B \), then \( A - C = B - C \) |
| 3. Multiplying both sides of an equality by the same number leaves the equality remaining true. 
  If \( A = B \), then \( AC = BC \) |
| 4. Dividing both sides of an equality by the same number leaves the equality remaining true. 
  If \( A = B \), then \( \frac{A}{C} = \frac{B}{C} \) provided that \( C \neq 0 \) |
| 5. Interchanging the two sides of an equality leaves it remaining true. 
  If \( A = B \), then \( B = A \) |

\[ C \neq 0 \] means that \( C \) is not equal to zero.
Now we will solve equations, using the properties of equalities.

Example 3
Let’s solve the equation \( x + 9 = 4 \).
To reduce the left hand side to \( x \) alone,
subtract 9 from both sides:

\[
x + 9 - 9 = 4 - 9
\]
Therefore \( x = -5 \).

Check 3
Check that \(-5\) is the solution to the equation \( x + 9 = 4 \) by substituting \(-5\) for \( x \).

Problem 4
Check that \(-5\) is the solution to the equation \( x + 9 = 4 \) by substituting \(-5\) for \( x \).

Check 3
Solve the equation \( x + 4 = 12 \).

Problem 5
Solve the following equations:

1. \( 10 + x = 7 \)
2. \( y - 7 = 6 \)

Example 4
Solve the equation \( 4x = 6 \).
How can we convert this to the form \( x = \) ?

Answer
You could also multiply by \( \frac{1}{4} \), the reciprocal of \( 4 \) on the LHS, to make the coefficient of \( x \) equal to \( 1 \).

Check 4
Solve the equation \( 2x = -16 \).

Problem 6
Solve the following equations:

1. \( -6x = 3 \)
2. \( \frac{1}{4} x = 5 \)
Ways of Solving Equations

We can solve the equation $5x = 6 + 4x$ in the following way.

Which properties of equalities are we using to get from equation (1) to equation (2)?

$$5x = 6 + 4x \quad \text{.................................. (1)}$$

To eliminate terms including $x$ from the RHS

$$5x - 4x = 6 + 4x - 4x \quad \text{.................................. (2)}$$

$$x = 6$$

If we calculate just the RHS of equation (2), we get the following:

$$5x - 4x = 6 \quad \text{.................. (3)}$$

When comparing equations (1) and (3) above, we can see that the $+4x$ term on the RHS of (1) has moved to the LHS, with the sign reversed, becoming $-4x$.

In general, we can move a term from one side of an equality to the other by reversing the sign. This is called **rearranging terms**.

Let's solve equations, using the idea of rearranging terms.

**Example 1**

1. Let's solve $x - 3 = 11$.

   Move the term $-3$ to the RHS:

   $$x = 11 + 3$$
   $$x = 14$$

2. Let's solve $4x = -2x + 18$.

   Move the term $-2x$ to the LHS:

   $$4x + 2x = 18$$
   $$6x = 18$$
   $$x = 3$$
Check 1
Solve the following equations:

1. \[ x + 6 = 9 \]
2. \[ 7x = 5x - 6 \]

Problem 1
Solve the following equations:

1. \[ 4x - 7 = -15 \]
2. \[ 1 - 2x = 11 \]
3. \[ 9x = 4 - 7x \]
4. \[ -2x = 3 - x \]

Example 2
Solve the equation \[ 9x - 5 = 2x + 23 \].

Hint
Move the terms including \( x \) to the LHS, and the number terms to the RHS.

Answer

\[
9x - 5 = 2x + 23 \\
9x - 2x = 23 + 5
\]

Check 2
Solve the following equations:

1. \[ 5x + 8 = 2x - 4 \]
2. \[ 2x + 7 = 19 - 4x \]

Problem 2
Solve the following equations:

1. \[ 9 - x = 2 + 6x \]
2. \[ 8x - 10 = 9x - 4 \]
3. \[ 2 + 10x = 4x - 1 \]
4. \[ -7x + 1 = -x + 1 \]

We can use the following procedure to solve equations:

1. Move the terms including \( x \) to the LHS and the number terms to the RHS.
2. Put in the form \( ax = b \).
3. Divide both sides by \( a \), the coefficient of \( x \).
Various Equations

Now we'll solve equations including parentheses, and with non-integer coefficients.

Example 1
Solve \( 3x - 2(x - 1) = 8 \).

Hint
Start by eliminating the parentheses.

Answer
Eliminating the parentheses:
\[
3x - 2(x - 1) = 8 \\
\Rightarrow 3x - 2x + 2 = 8 \\
\Rightarrow 3x - 2x = 8 - 2 \\
x = 6
\]
Answer \( x = 6 \)

Problem 1
Solve the following equations:

1. \( 1 + 2(x - 4) = 3 \)
2. \( 4(3x + 4) + 1 = -7 \)
3. \( 2x - 3(2 - x) = 4 \)
4. \( 7x + 9 = 3(x - 1) \)

What techniques can we use to solve equations when the coefficients are not integers?

Example 2
Solve \( 2.5x - 0.4 = 4.6 \).

Hint
To make the coefficients into integers, multiply both sides by 10.
\[
(2.5x - 0.4) \times 10 = 4.6 \times 10 \\
2.5x \times 10 - 0.4 \times 10 = 4.6 \times 10
\]

Problem 2
Solve the equation in Example 2.

For an equation including decimal fractions, multiply both sides by 10, 100, 1000, and so on, to convert the equation to a form with no decimal fractions, before solving.

Problem 3
Solve the following equations:

1. \( 0.5x - 0.3 = 4.2 \)
2. \( 0.7x + 1.47 = 1.5x - 0.93 \)
Example 3

Solve \( \frac{1}{3} x - 4 = \frac{1}{5} x \).

Hint

To convert the coefficients to integers, multiply both sides by 15, a common multiple of 3 and 5, the denominators.

Answer

\[ \frac{1}{3} x - 4 = \frac{1}{5} x \]

Multiply both sides by 15

\[ \left( \frac{1}{3} x - 4 \right) \times 15 = \frac{1}{5} x \times 15 \]

\[ \frac{1}{3} x \times 15 - 4 \times 15 = \frac{1}{5} x \times 15 \]

\[ 5x - 60 = 3x \]

\[ 5x - 3x = 60 \]

\[ 2x = 60 \]

\[ x = 30 \]

To solve an equation with coefficients including fractions, first multiply both sides by a common multiple of the denominators to convert it to a form without fractions. Converting the equation in this way is called canceling the denominator. If you can easily find the least common multiple of the denominators (the least common denominator), this is the simplest value to use.

Problem 4

Solve the following equations:

1. \( \frac{x}{4} - \frac{1}{2} = \frac{x}{2} + \frac{3}{4} \)
2. \( \frac{2x - 1}{3} = \frac{x + 3}{2} \)

With the equations we have studied so far, the terms can be rearranged into the form:

\[ \text{(linear expression)} = 0 \]

This form of equation is called a **linear equation**.
Equalities

1. Write an equality representing each relation among the following quantities.
   1) To buy five stamps at a yen each, I paid with a 1000 yen bill, and received change of b yen.
   2) When starting with 40 tangerines, and giving two each to x people, there were eight left.

Solutions of equations

2. Which of the following equations have 6 as a solution?
   1) \( x - 6 = 1 \)
   2) \( \frac{x}{6} = 1 \)
   3) \( 2x + 1 = 13 \)
   4) \( 2x + 3 = 3x \)

Methods of solving linear equations (using properties of equalities)

3. Solve the following equations:
   1) \( x + 6 = 2 \)
   2) \( x - 8 = -3 \)
   3) \( 4x = 24 \)
   4) \( \frac{1}{3}x = -2 \)

Methods of solving linear equations (by rearranging terms)

4. Solve the following equations:
   1) \( 2x - 3 = 7 \)
   2) \( 3x = 28 + 7x \)

5. Solve the following equations:
   1) \( 5x - 19 = -4x + 8 \)
   2) \( 2x + 12 = 7 - 3x \)
   3) \( 7x - 3 = 4x + 9 \)
   4) \( 5 - x = 5x + 8 \)
Akira and Tadashi have agreed to meet. Akira receives the following note from Tadashi.
What is the date of their meeting?

Let's meet on the day for which the sum of the date one week earlier and the date of the previous day equals 22.

Let's think about this problem, using an equation.

Suppose \( x \) is the date the two will meet

The date one week earlier is \( x - 7 \)
The date of the previous day is \( x - 1 \)

Therefore, since the sum of these two dates is 22, we obtain the following equation:

\[
(x - 7) + (x - 1) = 22
\]

Solving this equation gives \( x = 15 \), so the date they will meet is the 15th.

Check 1 Find the solution to the following problem, following the same four steps as 1 to 4 above.

In the first year of a certain junior high school, there are 154 pupils, including boys and girls, and the boys outnumber the girls by 18. Find how many girls there are.
Now we can solve all sorts of problems, using equations.

**Example**

I want to buy some oranges at 90 yen each and some apples, to make a total of 15 fruit and so that the total cost is 1800 yen. How many oranges and how many apples should I buy?

Suppose \( x \) is the number of oranges; then fill in the blanks in the following table.

<table>
<thead>
<tr>
<th></th>
<th>Price for one (yen)</th>
<th>Number bought</th>
<th>Cost (yen)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Orange</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Apple</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Hint: We can construct an equation from the following price relation.

\[(\text{Price of oranges}) + (\text{Price of apples}) = 1800 \text{ (yen)}\]

**Answer**

Problem 1

I bought ten lead pencils and five colored pencils totaling 1300 yen. The price of a colored pencil is 20 yen more than that of a lead pencil. Find the unit price of both the lead pencils and colored pencils.
If I try to share some origami among children, at four sheets each, there are nine sheets short. At three sheets each there are 15 sheets left over.
Find how many children there are, and how many sheets of origami.

From the following relations:

- Giving four sheets to each child there are nine sheets short
- Giving three sheets to each child there are 15 sheets left over

We can express the number of sheets of origami in two different ways to construct an equation. Supposing that \( x \) is the number of children, then according to (1), the number of sheets of origami is:
\[
(4x - 9) \text{ sheets.}
\]

Problem 2 In Example 2, answer the following questions:

1. For relation (2), show the number of sheets of origami, using \( x \).

2. Following the hint, construct an equation.

3. Solve the equation you constructed in question 2 to find the answer to Example 2.

Problem 3 I want to buy five cakes, but I am 150 yen short. If I buy four, I have 80 yen left over. Find the price of one cake, and how much money I have.
Example 3
Bert sets off from home to go to school. Four minutes later, his elder brother Arnold sets off for school, following him. If Bert walks at 50 meters per minute and Arnold walks at 70 meters per minute, how long will it take Arnold from the time of leaving home to catching up with Bert?

Hint If it takes $x$ minutes from the time Arnold leaves home until he catches up with his brother, we can illustrate the quantities in the problem by drawing diagrams and a table, as follows.

<table>
<thead>
<tr>
<th>Speed (m/min)</th>
<th>Time (min)</th>
<th>Distance traveled (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bert</td>
<td>50</td>
<td></td>
</tr>
<tr>
<td>Arnold</td>
<td>70</td>
<td>$x$</td>
</tr>
</tbody>
</table>

Arnold catches up with Bert when the following remains true:

$$\text{(Distance traveled by Bert)} = \text{(Distance traveled by Arnold)}$$

Use this fact to construct an equation.

Problem 4 In Example 3, answer the following questions:

1. Fill in the blanks in the table above.
2. Following the hint, construct an equation.
3. Solve the equation you constructed in question 2 to find the answer to Example 3.

In Example 3, we can draw a diagram as shown on the right to illustrate how the brothers walk.
Problem 5  Anne leaves home at 9 o’clock to go to the station. Her younger sister Barbara realizes that she has forgotten something, and sets out from home on her bicycle at 10 past 9 to chase after her sister.
If Anne walks at 60 meters per minute, and Barbara cycles at 210 meters per minute, how many minutes past nine is it when Barbara catches up with Anne?

Problem 6  In Problem 5, if the distance from home to the station is 800 meters, is the value you have found still the answer to the question?

Basic Exercises

Using linear equations (1)

I have 210 cm of tape. I am going to share the tape between Angela and her younger brother Brian so that Angela’s piece is 50 cm longer than Brian’s.

How long are the pieces of tape?

Think about the problem this way:
1  Suppose Brian’s piece is $x$ cm long.
2  Express the length of Angela’s piece using $x$.
3  Find an equality relationship and write this as an equation.
4  Solve the equation you constructed in step 3 to find the answer.

Using linear equations (2)

Cans of tea cost 120 yen, and cans of soft drinks cost 140 yen. I bought a total of eight cans for 1020 yen. How many cans of tea did I buy?
Chapter Summary Problems • A

1. Solve the following equations:
   1. \(-\frac{1}{2}x = 6\)
   2. \(5x - 1 = 9\)
   3. \(5 - 3x = -2x\)
   4. \(3x - 8 = 5x\)
   5. \(6x + 9 = 7x + 3\)
   6. \(5x - 6 = 2 - 3x\)
   7. \(4x + 7 = 4 - 2x\)
   8. \(-5x + 18 = 3x + 2\)
   9. \(3(2a - 5) = 6 - a\)
   10. \(2 - 4(x + 5) = 2x - 18\)

2. Solve the following equations:
   1. \(1.1x + 1.4 = 4.6 - 0.5x\)
   2. \(\frac{1}{2}x + 5 = \frac{1}{5}x - 1\)

3. In each of the following equations, when \(x\) has the value shown in brackets, what value must \(a\) have for the equation to remain true?
   1. \(3x + a = x - a\) \(\quad [1]\)
   2. \(3(x + 4) - 2a = 5\) \(\quad [-3]\)

4. There are 40 tangerines in box A, and eight in box B. If we transfer tangerines from box A to box B until box A holds twice as many tangerines as box B, how many do we need to transfer?

5. I went to buy notebooks at the stationery shop. With the amount of money I had, buying eight of the cheaper notebooks left me with 40 yen. But if I bought the notebooks that cost 60 yen more each, I could buy five notebooks and have only 10 yen left. Find how much cheaper each notebook costs, and how much money I had.
1. Solve the following equations:
   \[ 1.2(2x - 1) = 2.7x + 0.3 \]
   \[ \frac{3x + 5}{4} = \frac{x - 5}{3} \]

2. A cycle track goes around the perimeter of a lake. Alice cycles a lap at a speed of 15 km/h, and Bob cycles a lap at 10 km/h and takes exactly 10 minutes longer. What is the length of the track in kilometers?

3. Two tanks, A and B, are being filled with water; tank A at 4 liters per minute and tank B at 3 liters per minute. At exactly 10 o'clock the amount of water in the two tanks is measured, and tank A holds 160 liters, and tank B 60 liters.
   1. At \( x \) minutes after 10 o'clock, how many liters will be in each tank? Write an expression using \( x \).
   2. At what time will the amount of water in tank A be twice that in tank B?
   3. Will the amount of water in tank A reach three times the amount in tank B?

**Let's investigate!**

Think of a problem to fit the following equation:

\[ 2x + 70 = 150 \]
Finding the counterfeit coin

With skillful use of a balance or scales, we can find the counterfeit coin.

There are nine coins, and one of them is fake. Show how to find which one, using just two trials on a balance. We know that the fake coin is lighter than the real ones.

There are four bags full of coins. However, some of the bags contain only fake coins.
The weight of a real coin is 10 g, and the weight of a fake coin is 9 g.
Using the scales once only, show how to find all of the bags that hold fake coins.
In the next chapter — "Proportionality and Inverse Proportionality" you will use calculators for some problems. Before studying this section, take the time to learn these useful things you can do with a calculator.

**Multiplying or dividing repeatedly by the same number**

When finding the circumference of circles of different diameters, it is tedious to have to press \( 3.14 \) each time. In this case, with a calculator such as appears on the right, start by pressing:

\[
\begin{align*}
3.14 & \times
\end{align*}
\]

Then for diameters 2 cm, 3 cm, ... and so on, press:

\[
\begin{align*}
2 & = 6.28 \\
3 & = 9.42 \\
\vdots
\end{align*}
\]

The same technique also works for division.

For example, for various values of \( x \), when finding the value of \( \frac{1}{6} x \), press \( 6 \div \div \) then the \( x \) value and \( \div \div \) in turn.

Using the above technique, for various values of \( x \), calculate \( \frac{1}{6} x \) to two decimal places, and complete the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Value of ( \frac{1}{6} x )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Try pressing \( 4 \times \div \).

What numbers appear?
Also try numbers other than 4.
Chapter 4 Proportionality and Inverse Proportionality

1 Proportionality

Trigger

Alice's house has a bathtub which is a rectangular parallelepiped. She is filling the bathtub with water.

How long after starting to fill the bathtub should I turn off the water for it to be at the 50 cm level?

After 5 minutes

Supposing that the water continues flowing in at the same rate, let's show the relation between the time from starting to fill the bathtub and the water depth, using a diagram, a table, and a graph.

1 Diagram

Water depth (cm)

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>?</th>
</tr>
</thead>
<tbody>
<tr>
<td>40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2 Table

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>0</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth (cm)</td>
<td>0</td>
<td>10</td>
</tr>
</tbody>
</table>

3 Graph

Water depth (cm)

<table>
<thead>
<tr>
<th>Time (minutes)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>50</td>
</tr>
<tr>
<td>40</td>
</tr>
<tr>
<td>30</td>
</tr>
<tr>
<td>20</td>
</tr>
<tr>
<td>10</td>
</tr>
</tbody>
</table>

If you check the time, you don't need to keep going to look at the water level!
Bob's house has a bathtub that is not a rectangular parallelepiped, as shown below.

To fill the bathtub to the 50 cm level, can I just turn off the water after the same time as in Alice's house?

<table>
<thead>
<tr>
<th>Time (minutes)</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
</tr>
</thead>
<tbody>
<tr>
<td>Water depth (cm)</td>
<td>0</td>
<td>10</td>
<td>19</td>
<td>27</td>
<td>34</td>
</tr>
</tbody>
</table>

What difference is there in the way the water depth increases in the two bathtubs? Let's compare the change in the water level at different times after starting to fill the bathtub.
Proportional Quantities

Quantities Varying Together

In both of the examples shown on pages 86 and 87, the depth of the water and the time elapsed from starting to fill the bathtub are quantities that vary together. Let's compare the change in the water depth in the two bathtubs.

When filling the bathtub in Alice's house, if the water depth is \( y \) cm when \( x \) minutes have elapsed from starting to fill the bathtub, the relation between \( x \) and \( y \) is as shown in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>10</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
</tr>
</tbody>
</table>

Problem 1: In the table above, when the value of \( x \) is multiplied by 2, 3, or 4, what is the corresponding value of \( y \) multiplied by? If the value of \( x \) is reduced to \( 1/2 \), \( 1/3 \), or \( 1/4 \) of its previous value, what happens to the corresponding values of \( y \)?

When filling the bathtub in Bob's house, if the water depth is \( y \) cm when \( x \) minutes have elapsed from starting to fill the bathtub, the relation between \( x \) and \( y \) is as shown in the following table:

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>5</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>10</td>
<td>19</td>
<td>27</td>
<td>34</td>
<td>41</td>
</tr>
</tbody>
</table>

Problem 2: In the table above, when the value of \( x \) is multiplied by 2, 3, or 4, is the corresponding value of \( y \) also multiplied by 2, 3, or 4?

Whichever bathtub we are considering, determining the value of \( x \) also determines the corresponding value of \( y \). Letters used in the way that \( x \) and \( y \) are used here are called variables.
Example 1
When buying 80-yen stamps, the number of stamps determines the total cost.

Check 1
For the following quantities, what do we need to determine?

1. When reading a 180-page book, the number of pages left to read
2. The time taken for a car to reach a destination 100 km away

Proportional Quantities

When filling the bathtub in Alice's house, if we set a time since starting to fill the bathtub, how can we find the depth of the water at this time?

When filling the bathtub in Alice's house, the following relation holds true:

\[(\text{Depth of water}) = 2 \times (\text{Time from starting to fill the bathtub})\]

If we let the depth of water \( y \) minutes after starting to fill the bathtub be \( y \) cm, then the relation between \( x \) and \( y \) is as follows:

\[y = 2x\]

Equations representing proportionality

When the relation between two variables \( x \) and \( y \) is as shown by the following equation, we say that \( y \) is proportional to \( x \).

A fixed number or a letter that represents it is called a constant.

In the equation of proportionality above, the letter \( a \) is a constant, called the constant of proportionality.

When \( x \neq 0 \), the value of \( \frac{y}{x} \) is a constant, equal to the constant of proportionality.
To investigate proportionality, let's show $y$ as an expression involving $x$.

**Example 2**

For a circle with diameter $x$ cm, suppose the perimeter has a length of $y$ cm.

Then if we represent $y$ by an expression involving $x$, we get:

$$y = \pi x$$

And $y$ is proportional to $x$.

The constant of proportionality is $\pi$, the ratio of the circumference of any circle to its diameter.

**Check 2**

For the following examples, show that $y$ is proportional to $x$.

Give the constant of proportionality in each case.

1. Walking at 4 km per hour for $x$ hours, I go a distance of $y$ km.

2. The area of a triangle with a base of 12 cm and a height of $x$ cm is $y$ cm$^2$.

**Problem 3**

A car can travel 480 km on 40 liters of gasoline. If we say that this car can travel $y$ km on $x$ liters of gasoline, answer the following questions:

1. How many kilometers can the car travel on one liter of gasoline?

2. Give $y$ as an expression involving $x$.

3. To travel 60 km, how many liters of gasoline are required?

As you can see from Problem 2 on page 88, when filling the bathtub in Bob's house, it is not possible to express the relation between $x$ and $y$ in the form $y = ax$. Therefore, $y$ is not proportional to $x$.
Let's extend the range of values taken by variables, to include negative numbers, and investigate the relation of proportionality.

A car is traveling along an expressway from west to east, at 80 km/hour. Suppose that \( x \) hours after the car passes point P, the car is at point \( y \) km from point P. If we take the direction to the east as positive, the relation between \( x \) and \( y \) is shown by the following equation:

\[
y = 80x
\]

In this case, think about the position of the car when the time is expressed as a signed number.

**Example 3**

Position 3 hours before passing point P

Since we can change “3 hours before” into “-3 hours”, by substituting -3 for \( x \) in \( y = 80x \), we get:

\[
y = 80 \times (-3) = -240
\]

This means that the car is at a position 240 km west of the point P.

**Problem 4**

For the situation described above, answer the following questions:

1. Find the value of \( y \) for each value of \( x \), and fill in the blanks in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
</tr>
<tr>
<td>( y )</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
<td>_</td>
</tr>
</tbody>
</table>

2. For \( x \) and \( y \) above, when \( x \) is negative, if the value of \( x \) is multiplied by 2, 3, or 4, what are the corresponding values of \( y \) multiplied by?

**Problem 5**

Water flows continuously into a tank, at 4 liters/minute. Taking 10 o'clock as a reference point, suppose that \( x \) minutes later the amount of water in the tank has increased by \( y \) liters.

1. Give an equation relating \( y \) to \( x \).

2. Find the value of \( y \) when \( x = -3 \). What do the values of \( x \) and \( y \) represent in this case?
Variable Ranges

Assuming that water flows into a tank at the same rate as in Problem 5 on the previous page, let's look at the way in which the values of x and y vary, from three minutes before until five minutes after the reference time of 10 o'clock.

If we are considering just the interval from three minutes before until five minutes after, the range of values for x is from −3 to +5 inclusive.

In this case, we can specify the range by saying that $x$ is greater than or equal to −3, which we can write $x \geq -3$, and also that $x$ is less than or equal to 5, which we can write $x \leq 5$.

We can also include the two inequalities in a single statement. Thus $x$ in the range −3 to 5 inclusive is written:

$$-3 \leq x \leq 5$$

Check 3 In the above example, as the variable $x$ ranges from −3 to 5 inclusive, the variable $y$ ranges from −12 to 20 inclusive. Write this range for $y$ using inequality symbols.

Example 4 The variable $x$ has a range of all values less than 6. In this case we can write the range of $x$ using an inequality symbol as:

$$x < 6$$

which means $x$ is strictly less than 6, excluding the value 6 itself.
Finding Equations of Proportionality

Given variables $x$ and $y$, let's see how to write an equation for $y$ in terms of $x$.

Example 5

Suppose $y$ is proportional to $x$, and when $x = 4$, $y = -12$.

1. Write an equation for $y$ in terms of $x$.
2. When $x = -2$, find the value of $y$.

Answer:

Check 4

Suppose $y$ is proportional to $x$, and when $x = -6$, $y = -3$.

1. Write an equation for $y$ in terms of $x$.
2. Find the values of $y$ when $x = 2$ and when $x = -8$.

Problem 6

A spring is such that when a weight of not more than 20 g is hung from it, the length extension is proportional to the weight attached. When a 5 g weight is attached to the spring, it extends by 2 cm. Taking $y$ cm as the extension when a weight of $x$ g is attached, answer the following questions.

1. Write an equation for $y$ in terms of $x$.
2. When an 18 g weight is attached, by how many centimeters does the spring extend?
3. The range of the variable $x$ is given by $0 \leq x \leq 20$. What is the corresponding range of $y$?
For the proportionality relation shown by \( y = 2x \), fill in the blanks in the following table, and use the values to draw a graph in the diagram on the left.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When drawing the graph in the Trigger question above, we know that:

When 2, \( y = 4 \)

So this fixes the position of the point shown in the diagram on the right.

If we extend the range of the variables to include negative numbers, how can we find a point on the graph?

To find a point when we include negative numbers, we can think of two number lines, intersecting at right angles at their origins, as in the diagram on the right.

In this diagram:

- The horizontal number line is called the **x-axis**, or horizontal axis.
- The vertical number line is called the **y-axis**, or vertical axis.
- The \( x \) - and \( y \)-axes together are called **coordinate axes**.
- The point \( O \) where the coordinate axes intersect is called the **origin**.
To show the position of point P in the diagram on the right, we draw straight lines from P at right angles to the x and y-axes, and read off the values at the points of intersection: 4 and 3. This we write (4, 3). In this case we say:

4 is the **x-coordinate** of point P.
3 is the **y-coordinate** of point P.
(4, 3) are the **coordinates** of point P.

We also sometimes write the point P as P (4, 3).

The coordinates of the origin are (0, 0).

The notation P (4, 3) shows that point P is at a position 4 to right and 3 upward from the origin.

---

**Check 1**
In the diagram on the right, give the coordinates of points A, B, C, D, E, and F.

**Check 2**
Draw the following points in the diagram on the right.

Q(3,4)  R(4, -3)
S(-2, -4)  T(-4, 1)
U(0, -2)  V(4, 0)

---

**Let's try!**

Look for cases in everyday life where we use two numbers together to indicate a position, like coordinates.

Where in Japan would this be?

---

1— Proportionality
Graphs of Proportionality

For $y = 2x$, find the value of $y$ corresponding to each value of $x$ in the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>...</th>
<th>-4</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>...</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The points with the $x$ and $y$ values from the above table as coordinates are shown in the diagram on the left.

Problem 1: For $y = 2x$, let $x$ take the values from -4 to 4 in steps of 0.5, and find the corresponding values of $y$. Add the points with these pairs of $x$ and $y$ values as coordinates to the diagram on the left.
As we take more and more points whose coordinates are sets of values \( x \) and \( y \) such that \( y = 2 \cdot x \) holds, eventually as in the diagram on the lower right on the previous page, the collection of these points forms a line. The line formed in this way is the graph of \( y = 2 \cdot x \). This straight line is the graph of the Trigger question on page 94, extended in a straight line to the lower left.

Check 1. Draw the graph of \( y = 3 \cdot x \).

Let's try drawing a graph when the constant of proportionality is negative.

Problem 2. Use the following procedure to draw the graph of \( y = -2 \cdot x \).

1. Find the value of \( y \) for each value of \( x \), and fill in the blanks in the following table.

2. Draw the points with the \( x \) and \( y \) values from the above table as coordinates in the diagram on the lower right on the previous page.

3. Draw the graph of \( y = -2 \cdot x \).

Problem 3. Draw graphs showing the following proportionality relations.

1. \( y = x \)

2. \( y = -\frac{1}{2} \cdot x \)
In the relation \( y = ax \), when the value of \( x \) increases by a certain amount, let's investigate the change in the value of \( y \).

Problem 4

For \( y = 2x \), answer the following questions.

1. When \( x \) increases, does \( y \) increase? Or does it decrease?
2. If \( x \) increases by 1, by how much does \( y \) change?

Problem 5

Similarly for \( y = -2x \), answer the two questions in Problem 4 above.

Problem 6

In a graph of proportionality, what is the difference if the constant of proportionality is positive or negative? What can you say that is true of both cases?

Graph of \( y = ax \)

The graph of \( y = ax \) is a straight line through the origin.

<table>
<thead>
<tr>
<th>When ( a &gt; 0 )</th>
<th>When ( a &lt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Increase</td>
<td>Increase</td>
</tr>
<tr>
<td>Increase</td>
<td>Decrease</td>
</tr>
</tbody>
</table>

Problem 7

In the diagram on the right, \( \text{①} \) and \( \text{②} \) are graphs of proportionality. For each, give an equation for \( y \) in terms of \( x \).

Notice that graph \( \text{①} \) goes through \( (3, 2) \).
1. **Quantities varying together**
   You have to give the same number of sheets of paper to each of five children. To determine how many sheets you need in total, what quantity do you need to decide?

2. **Proportional quantities**
   A rectangular window has a height of 60 cm; when it is opened by $x$ cm, the area of the opening is $y \text{ cm}^2$. Express $y$ in terms of $x$, and show that $y$ is proportional to $x$. Find the constant of proportionality.

3. **Variable ranges**
   In Exercise 2, the variable $x$ takes values from 0 to 50 inclusive. Express the range of $x$ using inequality symbols.

4. **Finding equations of proportionality**
   $y$ is proportional to $x$, and when $x = 2$, $y = -8$.
   1. Find an equation showing $y$ in terms of $x$.
   2. Find the values of $y$ when $x = 3$, and when $x = -6$.

5. **Coordinates**
   Give the coordinates of point A in the diagram on the right.

6. **Draw point B ($-3, 1$) in the diagram on the right.**

7. **Graphs of Proportionality**
   Draw the graph of $y = \frac{3}{2}x$. 

---

1 — Proportionality 99
2 Inverse Proportionality

1 Inversely Proportional Quantities

In the diagrams below, we'll try drawing rectangles ① and ② with one vertex at the point A.

For the rectangles in cases ① and ②, we will find the height of the rectangle when the width is 1 cm, 2 cm, 3 cm, and so on, and fill in the blanks in the tables below.

1 Rectangles with an area of 18 cm²

| Width (cm) | 1 | 2 | 3 | 4 | 5 | 6 | ...
|------------|---|---|---|---|---|---|-----
| Height (cm)|   |   |   |   |   |   |     |

2 Rectangles with a perimeter of 18 cm

| Width (cm) | 1 | 2 | 3 | 4 | 5 | 6 | ...
|------------|---|---|---|---|---|---|-----
| Height (cm)|   |   |   |   |   |   |     |

(One graduation is 1 cm)
In both cases ① and ② of the Trigger question on the previous page, as the width of the rectangle increases, the height decreases. Let’s look more closely at how the width and height of the rectangles change in both cases.

**Problem 1** In case ①, as the width of the rectangle is multiplied by 2, 3, or 4, how does the height change? What about the rectangles in case ②?

In case ①, as the width of the rectangle is multiplied by 2, 3, 4, and so on, the height is multiplied by 1/2, 1/3, 1/4, and so on.

Let’s write the relation between the width and height of the rectangle in case ① using an equation.

For a rectangle with an area of 18 cm², if the width is \( x \) cm, and the height is \( y \) cm, then the relation between \( x \) and \( y \) is given by the following equation:

\[
x y = 18
\]

So we can write an equation for \( y \) in terms of \( x \) as follows:

\[
y = \frac{18}{x}
\]

**Equations representing inverse proportionality**

When the relation between two variables \( x \) and \( y \) is as shown by the following equation, we say that \( y \) is inversely proportional to \( x \).

For inverse proportionality too, the constant \( a \) above is called the **constant of proportionality**. When \( y \) is inversely proportional to \( x \), the product of \( x \) and \( y \) is a constant value, equal to the constant of proportionality.
Let's show examples of inverse proportionality as equations for \( y \) in terms of \( x \).

**Example 1**
A car travels a distance of 150 km for \( y \) hours at a speed of \( x \) km/hour.
Here, the equation for \( y \) in terms of \( x \) is:
\[ y = \frac{150}{x} \]
Where \( y \) is inversely proportional to \( x \).
The constant of proportionality is 150, representing the distance traveled.

**Check 1**
In the two following cases, show that \( y \) is inversely proportional to \( x \). Also find the constant of proportionality, and say what this constant represents.

1. When a 120 cm length of string is cut into \( x \) equal lengths, the length of each piece is \( y \) cm.
2. A triangle of area 8 cm\(^2\) has a base of length \( x \) cm and height \( y \) cm.

**Note**
1. In some cases, such as \( x \) in (1) above, the variable may take only natural numbers as values.

**Problem 2**
There are 4 liters of fuel. Supposing that if this fuel is used at the rate of \( x \) liters per hour, it will last for \( y \) hours, answer the following questions.

1. Write an equation for \( y \) in terms of \( x \).
2. If the fuel is used at the rate of 0.2 liters per hour, how many hours will it last?

**Problem 3**
Considering the rectangles in case \( @ \) on page 100, with a width of \( x \) cm and a height of \( y \) cm, write an equation for \( y \) in terms of \( x \). Can we say that \( y \) is inversely proportional to \( x \)?
Finding Equations for Inverse Proportionality

Given a set of \( x \) and \( y \) values, let's see how to write an equation for \( y \) in terms of \( x \).

Example 2

Suppose \( y \) is inversely proportional to \( x \), and when \( x = 4 \), \( y = 3 \).

1. Write an equation for \( y \) in terms of \( x \).
2. Find the value of \( y \) when \( x = -2 \).

Answer

\[ y = \frac{12}{x} \]

Check 4

Suppose \( y \) is inversely proportional to \( x \), and when \( x = -6 \), \( y = 8 \).

1. Write an equation for \( y \) in terms of \( x \).
2. Find the values of \( y \) when \( x = 4 \) and \( x = -2 \).

Problem 4

A tank takes 80 minutes to fill with water poured in at 3 liters per minute. If it takes \( y \) minutes to fill when water is poured in at \( x \) liters per minute, answer the following questions.

What is the total volume in liters of water held by the tank.

1. Write an equation for \( y \) in terms of \( x \).
2. If water is poured in at 5 liters per minute, how many minutes does it take for the tank to become full?
3. If the range of \( x \) is given by \( 4 \leq x \leq 10 \), find the corresponding range of \( y \).
Graphs of Inverse Proportionality

For \( y = \frac{6}{x} \), fill in the value of \( y \) for each value of \( x \) in the following table.

| \( x \) | ... | -6 | -5 | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | 4 | 5 | 6 | ...
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
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<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

If we plot the points with these sets of \( x \) and \( y \) values as coordinates, the result is as shown in the following diagram.

**Problem 1**

For \( y = \frac{6}{x} \), let \( x \) have values from 1 to 6 in steps of 0.5, and find the corresponding values of \( y \). Plot the points with these sets of \( x \) and \( y \) values as coordinates in the diagram above.

**Problem 2**

For \( y = \frac{6}{x} \), find the values of \( y \) when \( x = 0.5 \) and when \( x = 12 \).

Adding more and more points with coordinates satisfying \( y = \frac{6}{x} \) results in the set of points forming a curve, as shown in the upper right diagram on the next page. This curve is the graph of \( y = \frac{6}{x} \).

In an inverse proportionality relation, there is no value for \( y \) when \( x = 0 \).

4 — Proportionality and Inverse Proportionality
As the value of $x$ increases, what happens to the graph?

Check 1

Draw the graph $y = \frac{8}{x}$.

Let's try drawing the graph of a case where the constant of proportionality is negative.

Problem 3

Use the following procedure to draw the graph $y = -\frac{6}{x}$.

1. Find the value of $y$ for each value of $x$, and fill in the blanks in the following table.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$\cdots$</th>
<th>$-6$</th>
<th>$-5$</th>
<th>$-4$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$4$</th>
<th>$5$</th>
<th>$6$</th>
<th>$\cdots$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>$\cdots$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. Draw points with the $x$ and $y$ values from the above table as coordinates in the top diagram on the right.

3. Draw the graph $y = -\frac{6}{x}$.
Let's look at the characteristics of a graph of inverse proportionality.

For a graph of inverse proportionality, what distinguishes the cases when the constant of proportionality is positive and negative? What can we say that applies to both cases?

Problem 4  
For \( y = \frac{6}{x} \), answer the following questions.

1. Find the values of \( y \) when the value of \( x \) is 10, 100, and 1000. As the value of \( x \) gets larger, what happens to the graph?

2. Let \( x \) take the values 0.1, 0.01, 0.001, ... getting closer and closer to zero. What happens to the graph in this case?

If \( \alpha \) is a constant, then the graph \( y = \frac{\alpha}{x} \) consists of two parts, which are smooth curves. The whole graph is called a hyperbola.

This graph does not intersect the \( x \) or \( y \) axis.

- Graph of \( y = \frac{a}{x} \)

The graph of \( y = \frac{a}{x} \) is a curve (in two parts) called a hyperbola.

When \( a > 0 \) 

When \( a < 0 \)
Basic exercises

Inverse proportional quantities

1 Suppose that a parallelogram with an area of 16 cm² has a base of \(x\) cm, and a height of \(y\) cm. Write an equation for \(y\) in terms of \(x\), and show that \(y\) is inversely proportional to \(x\). Find the constant of proportionality.

Finding equations for inverse proportionality

2 Suppose \(y\) is inversely proportional to \(x\), and when \(x = 3, y = -8\)
   1 Write an equation for \(y\) in terms of \(x\).
   2 Find the values of \(y\) when \(x = 2\), and when \(x = -4\).

Graphs of inverse proportionality

3 Draw the graph of \(y = \frac{12}{x^2}\).
We'll look at problems in everyday life to which we can apply proportionality and inverse proportionality.

As shown in the photograph on the right, we have a large number of identical nails. We want to find the number of nails. Since the weight of the nails is approximately proportional to the number of nails, we can find the number of nails by weighing them.

The total weight of the nails shown in the photograph of Example 1 is 180 g. Weighing fifteen of the same nails gave a weight of 27 g. Can you work out how many nails there are in total in the photograph of Example 1?

<table>
<thead>
<tr>
<th>Number of nails</th>
<th>15</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight (g)</td>
<td>27</td>
</tr>
</tbody>
</table>

I made a dinosaur out of wire, as shown in the photograph below. The weight of the dinosaur is 80 g, and I found that 3 meters of the same wire weighed 48 g. Find the total length of wire used to make the dinosaur.
Five people are going to fold 1000 origami cranes. But still the number to fold per person is too high, and so they decide to make each person's quota for folding equal to one-fourth of the number when there are five people. In this case, how many people are required to do the folding?

We can solve the above problem using inverse proportionality, as follows.

Since we can represent the number of cranes folded by each person as \( \frac{1000}{\text{number of people}} \), the number of cranes folded by each person is inversely proportional to the number of people. Therefore to make the number folded per person one-fourth of its value, we multiply the number of people by 4.

\[ 5 \times 4 = 20 \]

Answer: 20 people are needed for folding.

Problem 3

In the Trigger question, find the number of cranes folded per person for the cases of five people and twenty people, and check that the number to be folded with twenty people is indeed one-fourth of the number with five people.

Problem 4

Five pipes are arranged over a tank, so that each pipe can deliver the same amount of water per hour into the tank. To fill the tank with one pipe takes one hour. How many minutes does it take to fill the tank using three of the pipes? How about using five pipes? What quantities are linked by inverse proportionality?
We'll use graphs of proportionality to work out the following problems.

**Example 2**
Arnold and his younger sister Betty leave home at the same time, to go to the library 600 meters away. If Betty walks at 50 meters per minute, and her brother at 75 meters per minute, how many minutes will it take from leaving home until they are 150 meters apart?

**Hint** Suppose that \( x \) minutes after leaving home, Arnold and Betty are \( y \) m apart, then we can draw the following graphs showing how they walk. We can find the distance between them from the following expression:

\[ \text{(Distance walked by Arnold)} - \text{(Distance walked by Betty)} \]

Therefore, we can simply read off the \( x \) coordinate value at the point where the difference between the \( y \) coordinates is 150.

Problem 5 Find the answer to Example 2.

Problem 6 Using the graphs of Example 2, answer the following questions.

1. How many minutes will it take from leaving home for Arnold to reach the library?
2. When Arnold reaches the library, how many more meters will Betty have to walk to reach the library?

*Let's try*
In addition to the way we solved Example 2 and Problem 6, find other ways of reading off values from the graphs.
Chapter summary problems • A

1 For the following cases (①) to (③), write an equation for \( y \) in terms of \( x \).
Say in which cases \( y \) is proportional to \( x \), and in which cases inversely proportional to \( x \), and state the constants of proportionality.

① After using a length of \( x \) cm of a pencil of length 18 cm, the remaining length is \( y \) cm.

② When water is added to a container that holds 30 liters at the rate of \( x \) liters per minute, it takes \( y \) minutes to become full.

③ If I walk at 80 meters per minute, \( x \) minutes after leaving the starting point, I have gone \( y \) meters.

2 The following are graphs of proportionality and inverse proportionality. Find the constant of proportionality in each case, and write an equation for \( y \) in terms of \( x \).

3 Two variables \( x \) and \( y \) take values as shown in the following tables. For each of cases ① and ②, write an equation for \( y \) in terms of \( x \), and find the values that fit in the blanks in the tables.

1 When \( y \) is proportional to \( x \\
| x | 1 3 6 9 \\
| y | 4 -2

2 When \( y \) is inversely proportional to \( x \\
| x | 1 3 6 9 \\
| y | 4 -2

Answers Supplementary Exercises

Chapter summary problems 111
1. In rectangle ABCD as shown on the right, point P moves along the side BC from B to C. If BP is \( x \) cm, and the area of the triangle ABP is \( y \) cm\(^2\), write an equation for \( y \) in terms of \( x \), and draw the corresponding graph.

2. Some wire is such that 4 m weighs 160 g, and the price per 100 g is 120 yen. If the cost of \( x \) meters of this wire is \( y \) yen, write an equation for \( y \) in terms of \( x \).

3. If a rectangle has width \( x \) cm, height \( y \) cm, and area \( S \) cm\(^2\), the area \( S \) is shown as \( S = xy \).

   1. If the value of \( y \) is fixed at 4, what is the relation between \( x \) and \( S \)?

   2. If the value of \( S \) is fixed at 24, what is the relation between \( x \) and \( y \)?

Let's investigate!

By using proportionality, as in the example of the bath on page 86 we can make predictions, or as in the example of the nails on page 108 we can find ways to count things that would otherwise be difficult to count.

Look for other similar examples from everyday life.

Higashi-Meihan Expressway (Aichi Prefecture)
Window on math ... finding the area of a complicated shape

Start with a map of Fukui Prefecture and some thick card larger than the map. Now we'll find an approximation to the area of Fukui Prefecture.

1. Using the scale of the map, work out the area on the ground in square kilometers that the area of the whole sheet of card would represent.
2. Weigh the sheet of card.
3. Trace the outline of Fukui Prefecture from the map onto the card, cut out the shape, and weigh it.
4. Using the results of 1 to 3, calculate the area of Fukui Prefecture.

Trace a map of the prefecture you live in, and follow the above procedure. Compare the result you get with the actual area.

The method above is using a proportionality relation. Have you seen which quantities are proportional to each other?
Chapter 5 Plane Figures

1 Symmetrical Figures

By folding and cutting a square sheet of paper, we'll make the shapes shown in the following diagrams.

Try different ways of folding and cutting to get a variety of shapes.

By folding a square of paper exactly in half, cutting away a part, and opening it up, we get the same shape on each side of the fold.
If we fold a square of paper in the four ways shown in (a) to (d) below, how should we cut each one to make the shape shown in ② on the previous page? Draw in the cutting lines in each case.

(a)  

(b)  

c  

d  

To make the shape in ② by folding each of (c) and (d) one more time, where should we make the extra fold, and how should we cut? Try this with a real piece of paper to check.
Line Symmetry and Point Symmetry

Line Symmetry

As shown in the diagram on the right, when we make a fold on straight line \( l \) and the two parts of the figure match exactly, we say that the plane figure has **line symmetry**. The line of the fold is called the **axis of symmetry**.

In the figure on the left, draw in each axis of symmetry.

It looks as though there are lots of axes of symmetry...

Problem 1

Problem 2

From the following figures \( a \) to \( d \), select those that have line symmetry.

\[ a \quad b \quad c \quad d \]

Symbol of Fukuoka City  Symbol of Nirasaki City

Point Symmetry

**Trigger** What are the properties of figures \( b \) and \( d \) above?

Figures \( b \) and \( d \) remain the same if we turn them upside down. In other words, if we rotate them 180 degrees around a particular point, we get a shape exactly matching the original.

5 — Plane Figures

Try flipping the pages from page 140, watching the top left corner.
We say that a plane figure such as a parallelogram, which exactly matches itself rotated 180 degrees, has **point symmetry**. The point around which the figure rotates is called the **center of symmetry**.

---

**Congruent Figures**

If we cut a figure such as an isosceles triangle about the axis of symmetry, what is the relation between the two resulting parts?

If we can take two figures, and by rotating and turning over, we can make them match exactly, we call the figures **congruent**.

Any figure with line symmetry can be cut by the axis of symmetry into two congruent halves.

---

**Problem 3**

If we take a figure such as a parallelogram with point symmetry, to cut it with a straight line into two congruent parts, what straight line can we use?

---

**Let’s try!**

Look for objects around you that have shapes with line symmetry or point symmetry.

---

Cherry flower
Properties of Symmetrical Figures

Properties of Figures with Line Symmetry

In the diagram on the left, mark the point that matches point P when the isosceles triangle ABC is folded about the axis of symmetry c.

When points match up in this way when a figure with line symmetry is folded about the axis of symmetry, we say that they are corresponding points.

Problem 1
In the diagram for the Trigger question above, find more pairs of corresponding points, and draw a straight line joining each pair. Investigate how these lines are related to the axis of symmetry.

If a straight line passes through points A and B, the part between A and B is called line segment AB.

If we say "(straight) line AB" this normally refers to the straight line extending indefinitely in both directions.

If we extend the line segment AB indefinitely from B only, we call this half line AB.

If the length of line segment AB is the same as the length of line segment AC, we write AB = AC.

If line segment BC is at right angles to line segment AM, we say that these line segments are perpendicular, written BC \perp AM.
For the two corresponding points P and Q, in a figure with line symmetry as shown in the diagram on the right, and R is the intersection of line segment PQ and the axis of symmetry AM, then the following relations hold:

1. \( PR = QR \)
2. \( PQ \perp AM \)

**Properties of a figure with line symmetry**

In a figure with line symmetry, the axis of symmetry intersects the line segment joining any pair of corresponding points at right angles, and divides the line segment into two equal halves.

The point that divides a line segment into two equal halves is called the **midpoint** of the line segment. We also say that the midpoint **bisects** the line segment.

**Problem 2**

In a figure with line symmetry, where does the midpoint of a line segment join a pair of corresponding points?

In a figure with line symmetry, the lengths of corresponding line segments, and corresponding angles are equal.

The two half lines OA and OB radiating from the point O form an angle. We write this angle symbolically using \( \angle \) as \( \angle AOB \), pronounced “angle AOB”.

In the isosceles triangle ABC above, \( \angle ABC \) and \( \angle ACB \) are equal. We write this as \( \angle ABC = \angle ACB \).

**Check 1**

The diagram on the right shows a figure with line symmetry, where PR is the axis of symmetry.

1. Write the relation between line segment QS and straight line PR, using symbols.
2. Find equal line segments and equal angles, and use symbols to represent these equalities.
Properties of Figures with Point Symmetry

In a figure with point symmetry, points which match exactly when a figure is rotated 180 degrees are called corresponding points.

What is the relation between the center of symmetry of a parallelogram and a pair of corresponding points? Look at a number of pairs of corresponding points in the diagram on the left, and investigate their relation.

If R and S are a pair of corresponding points in a figure with point symmetry, then the following holds true:

1. Line segment RS passes through the center of symmetry O.
2. OR = OS

Properties of a figure with point symmetry

In a figure with point symmetry, the line segment connecting corresponding points passes through the center of symmetry, and is bisected by the center of symmetry.

Problem 3

In a figure with point symmetry, where is the midpoint of a line segment joining a pair of corresponding points located?

In a figure with point symmetry, the lengths of corresponding line segments, and corresponding angles are equal.

Check 2

The diagram on the right shows parallelogram ABCD, with a straight line PQ passing through the center of symmetry O. Find equal line segments and angles, and write expressions using symbols to show that they are equal.
The diagram on the right shows part of a figure with point symmetry, with O as the center of symmetry. Complete the diagram, and investigate the relation between corresponding sides.

Corresponding line segments in a figure with point symmetry are parallel.

Parallel lines do not intersect each other. When two lines such as AB and CD are parallel, we write this using the symbol // as \( AB \parallel CD \).

**Check 3** For the parallelogram on the right, write expressions using the symbol // to show that opposite sides are parallel.

**Problem 4** Let's investigate which triangles and quadrilaterals have line symmetry and point symmetry.

Find which of the following figures have line symmetry, and draw in the axis (or axes) of symmetry. Then find which have point symmetry, and draw the center of symmetry.

a) Isosceles triangle  
b) Equilateral triangle  
c) Parallelogram  
d) Oblong  
e) Rhombus  
f) Square
With a pair of compasses, draw a circle on a piece of paper, and cut it out. Fold the circle exactly in half: what can you say about the line of the fold?

Any point on a circle is the same distance from the center, and this distance is called the radius (plural "radii") of the circle.

Note: The boundary of a circle (and its length) is called the "circumference".

If A and B are two points on the circumference of a circle, the part of the circumference between A and B is called arc AB, written \( \overline{AB} \).

The line segment joining two points on the circumference of a circle is called a chord, and if the endpoints are A and B, this is called Chord AB.

A circle has line symmetry with respect to any diameter.

Problem 1: Draw any chord on the circle on the left, and draw the axis of symmetry of the chord.

Problem 2: A circle also has point symmetry. Where is the center of symmetry?
If we take a circle of paper folded as in the Trigger question on the previous page, and fold exactly in half twice more, then where are the folds when we unfold the paper? Draw in the fold lines on the rightmost figure in the following diagram.

<table>
<thead>
<tr>
<th>Fold</th>
<th>Fold</th>
<th>Unfold</th>
</tr>
</thead>
</table>

The figure enclosed by an arc and the two radii passing through its endpoints is called a sector. The angle between the two radii is called the central angle. A sector has line symmetry.

Problem 3 Draw the axis of symmetry of the sector shown in the diagram on the right.

This shape is a sector too!

The lower diagram on the right is formed by dividing the circumference of a circle into eight equal portions, and joining adjacent points by straight lines. This figure has eight equal sides and eight equal angles, and is called a regular octagon.

Figures with three, four, five, and more sides formed by line segments are called polygons. If all sides are the same length, and all angles are equal, the figure is called a regular polygon. A regular polygon has line symmetry.

Problem 4 How many axes of line symmetry does a regular octagon have?
Properties of Intersecting Circles

Two circles may intersect as shown in the following diagram. In each diagram, draw in the axis or axes of symmetry.

1. Circles of different radii
2. Circles with the same radius

A pair of intersecting circles has a straight line passing through both centers as an axis of symmetry. Then if the circles have the same radius, the straight line passing through the points of intersection is also an axis of symmetry.

Problem 5

In the diagram on the right, two circles with centers at points A and B intersect at P and Q; the intersection of the line segments AB and PQ is the point M. Answer the following questions about the quadrilateral AQBP.

1. Which sides are equal to each other?
2. Which angle is equal to \( \angle PAB \)?
3. What is the relation between PQ and AB? Use symbols to write your answers.

Problem 6

In problem 5, what has to be the case for AM to equal BM?

Using what we have learned here, we can construct a line perpendicular to a given straight line, or construct a straight line dividing an angle into two equal parts (this is also called "bisection").
Basic exercises

Line symmetry and point symmetry

Which of the following letters have line symmetry, and which have point symmetry?

FHNPSUX

Representing angles, parallel lines, and perpendicular lines

The diagram below shows parts of figures. Complete the figures in cases 1, 2, 3.

Properties of symmetrical figures

When the original figure has point symmetry about the point O

When the original figure has line symmetry about the axis e

The diagram below shows parts of figures. Complete the figures in cases 1, 2, 3.

Symmetry?

When of the following letters have line symmetry, and which have point symmetry?
Starting from any line segment, the following diagram shows how to construct a regular hexagon with this line segment as one side, using a ruler and compasses.

1 2 3

Explain why it is possible to construct a regular hexagon as in the trigger question.

Be careful not to change the setting of the compasses.

We will now look at various drawing constructions using ruler and compasses.
We use a ruler to draw straight lines. Any two points are sufficient to determine a straight line, so we can draw a straight line with a ruler if we know the two points it passes through.

We use a compass to draw a circle. We can also use them to mark off equal distances, or to copy a line segment.

Up to this point, we have not made any rules about which tools you can use for drawing. From now on, when we talk of drawing constructions, we will use rulers and compasses only.

**Problem 2** Construct triangle ABC with the three sides AB, BC, and CA having the lengths shown in the diagram on the right.

We write “triangle ABC” using a $\triangle$ symbol as $\triangle ABC$.

**Check 1** Write the names of the triangles in the diagram on the right, using the $\triangle$ symbol.
More Constructions

Using properties of symmetrical figures we'll investigate various techniques of drawing construction.

Perpendiculars

A piece of paper as shown in the diagram on the right has been folded along the line \( \ell \). Suppose \( S \) is a point corresponding to \( P \). What is the relation between the lines \( PS \) and \( \ell \)?

If two lines are at right angles, we say that one line is perpendicular (with respect to the other line). A perpendicular is also sometimes called a normal.

To construct the perpendicular from a point not on a line to the line, using the properties of two intersecting circles on page 124, we can find another point that the perpendicular passes through.

The following diagram shows how to construct a line that passes through point \( P \) not on line \( \ell \), and is perpendicular to \( \ell \).
Using the method of Example 1, construct a perpendicular from point P to line $\ell$.

Here is another method of constructing a line that passes through point P not on line $\ell$, and is perpendicular to $\ell$.

1. With point P as center, draw an arc intersecting line $\ell$, and let the points of intersection be A and B.

2. With A and B as centers, draw arcs of the same radius, which intersect at point C.

3. Draw line PC.

Using the method of Example 2, construct a perpendicular from point P to line $\ell$.

Draw the triangle ABC as shown below, and construct a perpendicular from vertex A to side BC, and a perpendicular from vertex B to side CA.
Let's consider the distance from a point to a line.

**Trigger** Suppose \( \ell \) is a straight line, and \( P \) is a point not on line \( \ell \). If we consider the distance from \( P \) to each point on line \( \ell \), when is this distance the shortest?

P and \( P' \) are corresponding points using \( \ell \) as an axis of symmetry.

If we draw a perpendicular from point \( P \) not on line \( \ell \) to line \( \ell \), and let the intersection with \( \ell \) be point \( Q \), then we call the length of the line segment \( PQ \) the distance of \( P \) from \( \ell \).

**Problem 2** Which point from \( A \) to \( F \) in the diagram on the right is the shortest distance from line \( \ell \)? Which is the longest distance from line \( \ell \)?

There are two parallel lines, \( \ell \) and \( m \). In this case, whichever point we choose on line \( \ell \), the distance to line \( m \) is the same. We call this distance the distance between the parallel lines \( \ell \) and \( m \).
Perpendicular Bisector

In the diagram on the right, point A is on the very corner of the page. Draw in line segment AB, then fold the page so that A and B exactly coincide. What can we say about the fold line?

A line that passes through the midpoint of a line segment at right angles is called the perpendicular bisector of the line segment.

The perpendicular bisector of the line segment AB is the axis of symmetry of AB, and the points A and B are corresponding points.

By looking at the diagram on the right, work out how to construct a perpendicular bisector of a line segment.

We can construct the perpendicular bisector of the line segment AB as follows.

1. With A and B as centers, draw arcs of the same radius, and let their points of intersection be C and D.

2. Draw the line CD.

---

Example 3

1

2

A

B

C

D

---

Basic Drawing Construction
Check 3 Construct a perpendicular bisector of line segment AB in the diagram on the left.

If we take point P on the perpendicular bisector \( \ell \) of line segment AB, then \( PA = PB \). Any point that is the same distance from A and B lies on the perpendicular bisector of AB.

Problem 3: Find the point on line \( \ell \) that is the same distance from the two points A and B.
Bisector of an Angle

In the diagram on the right, fold the corner of the page, so that the sides OA and OB of \( \angle AOB \) exactly coincide. What is the relation between the angles on either side of the fold?

The half line that divides an angle into two equal halves is called the *bisector* of the angle.

The bisector of an angle is the axis of symmetry of the angle.

We can construct the bisector of \( \angle AOB \) as follows.

1. Draw an arc with its center at O, to intersect the two sides at C and D.
2. With C and D as centers draw arcs of the same radius, to intersect at point E.
3. Draw the half line OE.

The half line OE in the diagram above is the bisector of \( \angle AOB \).

Therefore: \( \angle AOE = \angle BOE = 1/2 \angle AOB \)

Check 4 Draw \( \angle AOB \), and construct its bisector.
Since the bisector of an angle is the axis of symmetry of the angle, from any point on the bisector, the distances to the two sides of the angle are equal. Also, any point within the angle that is the same distance from the two sides of the angle lies on the bisector.

**Problem 4**

Take point P on the bisector of $\angle AOB$ that you constructed in Check 4, and construct perpendiculars PC and PD from P to the two sides of the angle. Construct a circle with the center P and radius equal to PC and PD.

**Example 5**

To construct a line passing through point O on line AB and at right angles to it, think of $\angle AOB = 180^\circ$ and construct the bisector of this angle.

**Problem 5**

Try carrying out the construction described in Example 5.

**Example 6**

Let's construct a regular octagon, using angle bisection.

1. Draw a circle, and draw a line passing through the center.

2. Using angle bisection, divide the angle around the center of the circle into eight equal parts.

3. Connect the intersections of the lines you drew in steps 1 and 2 with the circumference of the circle, in sequence.

**Problem 6**

Try carrying out the construction described in Example 6.
Using Constructions

Tangents to a Circle

As shown in the diagram on the right, if we take a vertical line parallel to the line passing through the center of a circle and move it progressively outward, there is an exact point at which they meet. This point on the line is called a **tangent** to the circle, and we call the single point the **point of contact** (or sometimes the **point of tangency**).

A tangent to a circle is perpendicular to the radius passing through the point of contact.

Let's see how to construct a tangent, based on its properties.

**Example 1**
We can construct a tangent to the circle with a center O at point A on the circumference, by drawing the line passing through A and perpendicular to the line OA.

**Problem 1**
Try carrying out the construction described in Example 1.

Remember that we learned how to construct a perpendicular on the previous page.
Using Constructions

Look at these problems, using the properties of a perpendicular bisector or an angle bisector.

Example 2
The diagram on the right shows line segment AB and two half lines AC and BD. Using constructions, find point P that is the same distance from AC, AB, and BD.

Hint
Use the fact that if a point is the same distance from the two sides of an angle, it must lie on the bisector of the angle.

Problem 2
Find point P, using angle bisectors.

Problem 3
In the drawing you constructed in Problem 2, construct a circle with center P, with the distance from P to the line segment AB as radius.

Example 3
The diagram on the left shows a fragment of a bronze mirror. Assuming that the shape of the mirror is a circle, construct the circle.

Hint
The perpendicular bisector of a chord is an axis of symmetry of the circle, and passes through the center. Use this fact first to find the center of the circle.

Problem 4
Try carrying out the construction described in Example 3.
Basic exercises

1. Constructing a perpendicular
   In the diagram on the right, taking the side AB as the base, construct the height of △ ABC.

2. Constructing a perpendicular bisector
   In the diagram on the right, find the midpoint of the line segment AB by construction.

3. Constructing an angle bisector
   In the diagram on the right, construct the bisector of ∠ AOB.
1. Draw \( \triangle ABC \), and answer the following questions.

      - Extend line segment BC from C, and mark point D on the extended line so that \( BC = CD \). Draw the line segment joining A and D.
      - Construct the midpoint E of AB and the midpoint F of AD, and draw a line segment joining E and F.

   2. If AC and EF intersect at G, use your compasses to check that the lengths of AG and GC are the same.

2. Which of the following shapes have both line symmetry and point symmetry? Give all cases that apply.

   - Equilateral triangle
   - Square
   - Regular hexagon
   - Parallelogram
   - Trapezoid
   - Circle

3. The diagram on the right shows square ABCD. For each of the following cases, what is the line segment corresponding to AB?

   1. When considering line symmetry about the diagonal BD.
   2. When considering point symmetry about the point O.

4. In the diagram on the right, CD is a half line drawn from point C on the straight line AB.

   1. Construct the bisectors CE and CF of \( \angle ACD \) and \( \angle BCD \), respectively.
   2. How big is \( \angle ECF \)?
Chapter summary problems

1. In diagrams ① and ② on the right, if:
   \[ \angle AOB = \angle COD = 90^\circ \]
   then
   \[ \angle AOC = \angle BOD \]
   Why is this?

2. A and B are any two points.
   Construct the circle which passes through A and B, and has a radius equal to twice the length of the line segment AB.

3. Construct the following angles.
   
   \[ \begin{align*}
   1 & \quad 135^\circ \\
   2 & \quad 15^\circ \\
   3 & \quad 75^\circ 
   \end{align*} \]

Let's investigate!

We constructed a regular hexagon on page 126 and a regular octagon on page 134. Think about and investigate ways of constructing regular polygons and make a report.

Can you find other ways of constructing a regular hexagon?

Can you construct a regular pentagon? Try to find out...

Answers
A farmer is going to build a bridge across the river that runs through the meadows. Where should he position the bridge to make the path from A to B on the opposite side of the river as short as possible? The river is a constant width, and the bridge must cross the river at right angles to the banks.

Suppose the position of the bridge is represented by PQ. Then the above problem means finding the position of PQ so that the length:

\[ AP + PQ + QB \]

is a minimum.

Let's try varying the position of the bridge, to see which gives the shortest path from A to B.

In the initial problem, since the width of river PQ is constant, all we need to find is the position for the smallest value of \( AP + QB \).

So now we look at Fig. 2 in which the width of river PQ has been set to zero.

In Fig. 2, draw the shortest path from A to B.

Now go back to the original problem, and draw the answer in Fig. 1.
Folding regular polygons with origami

Using a square piece of origami paper, we can fold an equilateral triangle and a regular octagon. Try this for yourself. See if you can see why these procedures produce the correct shapes.

- Equilateral triangle
- Regular octagon
A cube has six congruent faces, each being a square. Is there a solid which has six congruent faces, with each one being an equilateral triangle?

Cut out the equilateral triangles with tabs from pages 209 and 211, and try to make a solid with six faces being congruent equilateral triangles.

Are there other solids with faces all being congruent equilateral triangles, in addition to the one you made in the experiment above? Try to make one, using the triangles with tabs. For any shape you make, count the number of faces, edges, and vertices.
Try using a paperclip on the tabs to fasten two adjacent edges together.

Try making solid shapes using squares, regular pentagons, and regular hexagons.
A shape such as a cube or a rectangular parallelepiped whose faces are all planes is called a **polyhedron** (plural: polyhedra). Polyhedra have Greek names according to the number of faces: a pentahedron has five faces, a hexahedron has six, and so on. Both a cube and a rectangular parallelepiped are examples of a hexahedron.

**Check 1** How many faces does a triangular prism have?

**Trigger** How many vertices (corners) does a cube have? Also investigate how many faces meet at each vertex.

The diagram on the right shows a tetrahedron (four faces), in which the faces are all congruent equilateral triangles, of which three meet at each vertex. This solid is called a regular tetrahedron.

If a polyhedron has the following two properties, and is convex, we call it a **regular polyhedron**.

1. All faces are congruent, regular polyhedra.
2. The same number of faces meet at each vertex.

There are five different regular polyhedra, as seen in the sketches on the following page:

- Regular tetrahedron
- Regular hexahedron (cube)
- Regular octahedron (eight faces)
- Regular dodecahedron (twelve faces)
- Regular icosahedron (twenty faces)

The cube is a regular hexahedron.
Check 2 Find the number of edges and the number of vertices of a regular icosahedron.

Problem 1 Find the number of edges and the number of faces that meet at each vertex of a regular octahedron.

Problem 2 The hexahedron whose faces are all equilateral triangles, as described on page 142, is not a regular polyhedron. Investigate the number of sides and faces meeting at each vertex, and explain this.

Let's try!

Look at the following facts about polyhedra, and see if you can explain them.

1. There is only one polyhedron whose faces are all squares.
2. There is only one polyhedron whose faces are all regular pentagons.
3. There are no polyhedrons whose faces are all hexagons.
Let's try to classify the following solids.

The shapes (a) and (c) in the Trigger question are called **prisms**. A prism having a triangle as a base or quadrilateral is called a triangular prism or quadrangular prism respectively. Similarly, a prism having a pentagon as a base is called a **pentagonal prism**, and so on.

A shape like (e) is called a cylinder.

**Problem 1** What do prisms and cylinders have in common? And what are the differences between them?

If the base of a prism is a regular polygon, then we call the prism a regular prism.

**Check 1** Investigate the shapes and numbers of faces of a regular hexagonal prism.

---

Solid Shapes

Trigger

Let's try to classify the following solids.

---

2

---

Where do we start looking?

---

The shapes (a) and (c) in the Trigger question are called **prisms**. A prism having a triangle as a base or quadrilateral is called a triangular prism or quadrangular prism respectively. Similarly, a prism having a pentagon as a base is called a **pentagonal prism**, and so on.

A shape like (e) is called a cylinder.

**Problem 1** What do prisms and cylinders have in common? And what are the differences between them?

If the base of a prism is a regular polygon, then we call the prism a regular prism.

**Check 1** Investigate the shapes and numbers of faces of a regular hexagonal prism.

---

Uozu submerged forest museum (Toyama Prefecture)
Shapes ⑤ and ⑥ in the Trigger question are called **pyramids**. A pyramid with a triangle base or quadrilateral base is called a **triangular pyramid** or **quadrangular pyramid** respectively. Similarly, a pyramid with a pentagon as a base is called a **pentagonal pyramid**, and so on.

**Problem 2** What can you say about the lateral faces of a pyramid?

If the base of a pyramid is a regular polygon, and the lateral faces are all congruent isosceles triangles, then this is called a regular pyramid.

**Check 2** What sort of pyramid is a regular tetrahedron?

A shape like ① in the Trigger question is called a **cone**.

A ferry terminal in the form of a cone (Kumamoto Prefecture)

**Problem 3** In the following two cases, name differences and similarities between the two shapes.

1. Cylinder and cone
2. Pyramid and cone

**Problem 4** Two regular quadrangular pyramids have bases with all sides the same length. If we match up the two bases, what solid shape will it be?
Try placing a model of a regular octahedron on the desk. What positional relation can you see between the bottom face and the top face?

Normally we think of a plane as extending indefinitely in all directions. Line $\ell$ passing through the points $A$ and $B$ in plane $P$, therefore we say line $\ell$ lies in plane $P$.

For line $\ell$, there are many planes, such as $P, Q, \text{ and } R$ in the diagram on the right. However, for any point not on line $\ell$, there is only one plane in which $\ell$ lies that includes the point.

Problem 1 If there are three points, not lying on a single line, do these points determine a unique plane?
If two planes in space do not intersect, we say that they are parallel planes. If a line does not intersect a plane, we say that the line and plane are parallel.

Check 1 In the diagram on the right, what face of the rectangular parallelepiped is parallel to face ABCD? What faces are parallel to line AD?

If P and Q are two parallel planes, and R is another plane that intersects them, then the lines of intersection ℓ and m lie in the single plane R and do not intersect; therefore ℓ \parallel m.

Within space, if two lines are not parallel and do not intersect, we call them skew lines.

Check 2 In the rectangular parallelepiped of Check 1, which edges are in a skew position with line AD?

Problem 2 In the rectangular parallelepiped in the diagram on the right, which edges are in a skew position with line AG?

In the diagram on the right, the line segment AG is called a diagonal of the rectangular parallelepiped. BH, CD, and DF are also diagonals.
As shown in the diagram on the right, using a set square, open the cover of the book to an angle of 60 degrees. Now open the book to 90 degrees.

The diagram on the right shows rectangle ABCD rotated about line $\ell$ as far as rectangle EFCD, and so lines AD and ED are both perpendicular to line $\ell$.

In this case, the rotation angle $\angle ADE$ is called the angle between the two planes ABCD and EFCD.

When the angle between planes P and Q is a right angle, we say the planes P and Q are perpendicular, written $P \perp Q$.

Check 3 The figure shown in the diagram on the right is a triangular prism formed by cutting a cube into equal halves.

1. What is the angle between face BCFE and face ACFD?

2. Which faces are perpendicular to face BCFE?

Problem 3 In the triangular prism of Check 3, which faces are perpendicular to face ABC?
We made an L-shape line ABC, out of wire as shown in the diagram on the right, and rotated it about using AB as the axis. As BC moves, what shape does it leave behind? And what relation does AB have to this shape?

Consider plane P and line \( \ell \) which are not inclined in any direction. In this case, line \( \ell \) is perpendicular to any line lying on the plane P and passing through O, the point of intersection with P. Here we say that line \( \ell \) is perpendicular to plane P.

Problem 4. To stand thin rod \( \ell \) perpendicular to plane P using set squares, what is the minimum number of set squares we need? And what is the best way to do this?

When line \( \ell \) intersects plane P, if \( \ell \) is perpendicular to two lines \( m \) and \( n \) in P that pass through the point of intersection O, then this is sufficient for \( \ell \) to be perpendicular to the plane P.
Check 4  The solid figure on the right is a triangular prism formed by splitting a rectangular parallelepiped into two. Which edge of this triangular prism is perpendicular to edge DF? Which surface is perpendicular to DF?

If the point of intersection of a perpendicular line dropped from a point A to a plane P is H, then the length of line segment AH is called the distance of point A from the plane P.

In the prism or cylinder on the right, the bases are parallel, and the distance from a point on one base to the other base is called the height of the prism or cylinder.

For a pyramid or cone, the distance from the vertex to the base is called the height.

Problem 5  We put three regular tetrahedra with sides all the same lengths on a desk, and place a sheet of plastic on top of the three tetrahedra. Show that in this case the sheet of plastic is parallel to the surface of the desk.
Moving Planes

In the diagram below, line segment AB moves in the direction of the arrow by exactly the length of the arrow, while staying parallel. When the line moves in this way, what shape does it make?

We can think of a cylinder or prism as a solid made by the base moving in a perpendicular direction. The traces left by the perimeter of the base is the lateral surface of the solid, and the distance moved is the height.

Check 1 If we move an equilateral triangle in the direction perpendicular to its plane, what three-dimensional shape does this make?
Solids of Revolution

If we take rectangle ABCD or the right-angled triangle ABC shown on the right, and rotate it through a whole turn about line \( t \) as axis, what solid shape does this trace out in each case?

If just the edge AB is rotated in the same way, what three-dimensional shape does this make?

We can think of a cylinder or a cone as the solid created by rotating a rectangle or right-angled triangle. In this case, the edge AB that sweeps out the lateral surface of the cylinder or cone is called the bosen (in Japanese) of the cylinder or cone.

Problem 1

If we cut a cylinder or cone along a plane including the axis of revolution, what shape is the cut? If we cut along a plane perpendicular to the axis of revolution, what shape is the cut?
A shape such as a cylinder or cone formed by rotating a plane figure about a straight line of axis is called a **solid of revolution**.

A sphere is a solid of revolution formed by rotating a half-circle about its diameter.

Cutting a solid of revolution along a plane including the axis results in a symmetrical plane figure that has the axis of revolution as its axis of symmetry.

**Problem 2**  
If we make a plane cut through a sphere, where is the cut of maximum size possible?

**Check 2**  
In the quadrilateral ABCD shown on the right, $\angle BCD$ and $\angle ADC$ are right angles. If this quadrilateral is rotated with the edge CD as an axis, draw a sketch of the resulting shape.

**Problem 3**  
For each of the solids of revolution 1 and 2 shown on the right, what shape do you think was rotated to form them?
Unfolding Solid Shapes

Find different shaped boxes around you, and cut them open to make a flat shape.

The Unfolding of a Prism or Cylinder

The following diagram shows how to unfold a triangular prism, with the lateral faces joined side by side. Here ABCD is a rectangle.

Check 1

Draw the unfolding of a square prism with a base of 3 cm and a height of 5 cm.
Shade faces that are parallel when the shape is folded in the same color.

The unfolding of a cylinder can be formed by a rectangle and two circles, as shown in the following diagram. The width of the rectangle is equal to the circumference of the circular base.
Check 2
A cylinder has a base with a radius of 5 cm. To draw an unfolding of this cylinder, what should the width of the lateral surface rectangle be in centimeters?

Problem 1
In the diagram below, a thread is wrapped around the lateral surface from A to B, so as to be as short as possible.

In this case, draw in the position of the thread on the unfolding diagram below.

The Unfolding of a Pyramid or Cone

The diagram on the right shows a square pyramid. Try cutting along the edges AB, AC, AD, and AE, and draw the resulting unfolded shape.

The result of drawing the unfolding of a square pyramid with the lateral faces joined together appears on the right.
Check 3. Draw the unfolding of a regular hexagonal pyramid having a base with sides the length of 3 cm, and the equal sides of the isosceles triangles forming the lateral faces with sides the length of 4 cm.

Having drawn the unfolding, try cutting it out, and folding it into the original shape.

Trigger: Consider a pyramid with a regular polygon as the base, and increase the number of sides of the base. If we draw it unfolding with the lateral faces joined together, what will this look like?

The unfolding of a cone consists of a circle for the base, and a sector of a circle (a fan shape) for the lateral surface. The length of the arc edge of the fan shape representing the lateral surface is equal to the circumference of the base.

Check 4: In the unfolding of the cone shown on the right, find the radius and arc length of the fan-shaped lateral surface.
In order to draw the unfolding of the cone in Check 4 on the previous page, what else do we need to know?

**Trigger**

Let's investigate the relation between the arc length of a sector and the central angle.

If we take two sectors of the same circle with the same central angle, then we can rotate one sector about the center of the circle, until it exactly coincides with the other sector. Therefore, two sectors of the same circle with the same central angle have the same arc lengths.

**Problem 2**

As shown in the diagram on the right, if we increase the central angle of a sector to double or three times its value, this also multiplies the arc length by two or three.

For sectors of the same circle, the arc length is proportional to the central angle.

**Example 1**

Let's find the central angle of the sector part of the unfolding of the cone in Check 4 on the previous page. The arc length of the sector is equal to the circumference of the base, that is, the circle with center O', which is $6\pi$ cm. Since the circumference of the circle with center O is $8\pi$ cm, the arc length is $\frac{6\pi}{8\pi} = \frac{3}{4}$ of the circumference of the circle. Since the arc length is proportional to the central angle, the central angle is $\frac{3}{4}$ of 360 degrees, or 270 degrees.
Problem 3  Draw the unfolding of the cone in Check 4 on page 158, then cut it out, and fold into the cone shape.

Check 5  To draw the unfolding of the cone shown in the diagram on the right, what should the central angle of the sector be?

Since the arc length is proportional to the central angle, for a sector with a radius of \( r \), and a central angle of \( a^\circ \), the arc length \( \ell \) is given by the following expression:

\[
\ell = 2\pi r \times \frac{a}{360}
\]

In Example 1 on the previous page and Check 5, you can find the central angle using this expression.

In the same way that we investigated the arc length on the previous page, we can see that the area of the sector is also proportional to the central angle.

Therefore the area of a sector with radius \( r \) and central angle \( a^\circ \) is given by the following expression:

\[
S = \pi r^2 \times \frac{a}{360}
\]

Check 6  Find the arc length and area of a sector with a radius of 6 cm and central angle of 60°.
Basic exercises

Parallel and perpendicular lines
and planes, skew lines

1. The diagram on the right shows a regular pentagonal prism.
   1. Which edges are parallel to plane AFGB?
   2. Which face is parallel to edge BC?
   3. How many edges are in a skew position with edge BC?
   4. Which faces are perpendicular to edge BG?

Solids of revolution

2. Draw a sketch of the solid generated by rotating the trapezoid shown in the diagram on the right about line $l$ as axis.

Unfolding a cylinder

3. A cylinder has a base with a radius of 8 cm. To draw an unfolding of this cylinder, is it sufficient to know the length of the lateral surface?

Unfolding a cone

4. Answer the following questions about the unfolding of the cone shown in the diagram on the right.
   1. Find the radius and arc length of the sector forming the lateral surface.
   2. Find the central angle of the sector forming the lateral surface in degrees.
The diagram on the right shows the unfolding of a triangular prism. Can you identify the parts which are the base and the lateral faces?

The entire area of the surface of a solid is called the **surface area**.

**Check 1** Find the area of a base, the lateral surface area, and the total surface area of the prism in the Trigger question above.

**Example 1** Find the surface area of a cylinder with a base of radius 5 cm and a height of 10 cm.

**Answer**

**Check 2** Find the surface area of a cylinder with a base of radius 10 cm and a height of 5 cm.
Let's consider the surface area of a cone.

Example 2

Find the surface area of the cone we looked at in Check 4 on page 158, with a base of radius 3 cm and a generator of length 4 cm.

Answer

From Example 1 on page 159, the central angle of the sector representing the lateral surface is 270 degrees, so the lateral surface area is:

$$\pi \times 4^2 \times \frac{270}{360} = 12\pi \quad \text{(1)}$$

The base area is:

$$\pi \times 3^2 = 9\pi$$

Therefore the total surface area is:

$$12\pi + 9\pi = 21\pi$$

Answer $21\pi \text{ cm}^2$

In Example 2, the arc length of the sector is $\frac{6\pi}{8\pi}$ of the circumference of the circle with center O. The area of a sector and the arc length are both proportional to the central angle, and therefore in expression (1) we can also replace $\frac{270}{360}$ by $\frac{6\pi}{8\pi}$.

Check 3

Find the surface area of a cone with a base of radius 3 cm and a slant height of length 9 cm.

Let's try!

In Example 2 above, to calculate the surface area of the cone, in place of $\frac{270}{360}$ in the expression (1) we can also use $\frac{\text{(radius of base)}}{\text{(length of generator)}}$. See if you can see why this also works.
If we take two triangular prisms whose bases are congruent right-angle triangles, and which have the same height, we can put them together as shown in the diagram on the right, to make a rectangular parallelepiped. Let's use this fact to calculate the volume of one triangular prism.

A quadrangular prism with a rectangular base is a rectangular parallelepiped, and therefore as shown on the right, if the base has sides of length $a$ and $b$, and the height is $h$, the volume $V$ is given by:

$$V = abh$$

Here $ab$ is the area of the base of the prism, so if we let this be $S$, we have:

$$V = Sh$$

In this way, the volume of a prism or cylinder is given by:

$$(\text{base area}) \times (\text{height})$$

Check 1 Find the volume of the following solid shapes.

1

2
Volume of a Pyramid or Cone

If we draw all the diagonals of a cube, they intersect at a single point 0. Then we can think of the cube as made from six regular square pyramids. From this we can find the volume of a square pyramid with base of side 10 cm and height 5 cm.

In fact the volume of a cone or pyramid is always \( \frac{1}{3} \) the volume of the corresponding cylinder or prism with the same base area and height.

If the base area of a cone or pyramid is \( S \) and the height is \( h \), we can find volume \( V \) from the following formula:

\[
V = \frac{1}{3} S h
\]
Check 2 Find the volume of the following solid shapes.

1. Height 7 cm
2. Height 9 cm

Problem 1 If the radius of the base of a cone is $r$ and the height is $h$, find an expression for volume $V$ of the cone.

Window on math ... finding the area of a sector

We can write area $S$ of a sector of radius $r$ with arc length $\ell$ as

$$S = \frac{1}{2} \ell r.$$  

1. Look at the following diagram, and see if you can use it to explain the expression above.

2. Using the expression above, look again at Check 3 on page 163.
Basic exercises

1. **Surface area of a prism or cylinder**
   Find the surface area of the following solids.
   1. 
   2. 

2. **Surface area of a cone**
   For the cone shown in the diagram on the right, answer the following questions.
   1. Find the lateral surface area.
   2. Find the base area.

3. **Volume of a prism or cylinder**
   Find the volume of the solids in question 1.

4. **Volume of a cone**
   Find the volume of the following solids.
   1. 
   2. 

---

3—Surface Area and Volume of Solids 167
Chapter summary problems • A

1 The diagram on the right shows a triangular prism with a base that is a right-angled triangle. Find the answers that fit the following five descriptions.

1. Edges parallel to edge AD
2. Edges parallel to face DEF
3. Edges perpendicular to face DEF
4. Faces perpendicular to face ADEB
5. Edges in a skew position with edge AD

2 A regular square prism has a base with a side of 3 cm, and the isosceles triangles forming the lateral faces has equal sides of length 5 cm. Draw an unfolding of this pyramid.

3 Find the surface area and volume of the following solids.
   1. A cylinder with a base of radius 3 cm and a height of 5 cm
   2. The square pyramid in the diagram on the right

4 The right-angled triangle ABC in the diagram on the right is rotated about edge AC as axis to form a solid of revolution. Answer the following questions about this solid of revolution.
   1. Draw a sketch of the solid.
   2. Find the volume.
Chapter summary problems B

1. In the following statements (1) to (3), if we replace “line” by “plane” and replace “plane” by “line”, are the statements still true?

   1. If two lines are each parallel to a third line, then they are parallel to each other.
   2. If two lines are each perpendicular to a plane, then they are parallel to each other.
   3. If two planes are parallel to each other, then a line that is perpendicular to one plane is also perpendicular to the other plane.

2. As shown in the diagram on the right, by joining the intersections of the diagonals of each face of a cube, we can construct a regular octahedron. If the length of an edge of the cube is 6 cm, what is the volume of the octahedron?

3. As shown in the diagram on the right, a cone has a base of radius 8 cm, and is rolled on a plane about vertex O as its center. In order to complete rolling exactly once around the path shown by the red line, the cone has rotated through just two whole turns. Find the length of the generator of this cone, and the surface area.

Let's investigate!

If we take a regular square pyramid whose edges are all the same length, and a regular tetrahedron, and join them together by exactly aligning two faces together, how many faces does the resulting polyhedron have?

Answers
Projections and Sections

Projections

What solid looks like a circle seen from above, and an isosceles triangle seen from the side?

To represent a solid object on a plane, we can use a sketch or an unfolding, but in addition we can draw the shape of the object seen from a particular direction.

A drawing of the shape of an object seen from a particular direction is called a projection. Often we draw a projection of an object seen from above together with a projection seen from the side. For example, the following shows a projection drawing of a triangular prism.

Problem 1

For each of the projection drawings 1 to 3, which does it represent, from the following: a rectangular parallelepiped, a triangular pyramid, a quadrangular pyramid, a cylinder, a cone, or a sphere?

1 2 3
Sections

If we make a plane cut through a cube, the cut face (called a section) can take various shapes. Diagrams (1) and (2) below show some of the sections that result from making a plane cut through a cube.

(1)  

(2)

Problem 2:  
In the cuts shown in diagram (1), which of the following does not appear: isosceles triangle, equilateral triangle, trapezoid, square, or oblong?

Problem 3:  
In the cuts shown in diagram (2), there is a case in which the section is a regular hexagon. How do you cut the cube to make this shape? Draw the answer on the cube shown on the right.

Problem 4:  
Diagram 1 on the right shows projections of a solid shape that results from cutting a cube along a particular plane. Diagram 2 shows a partial perspective sketch of the resulting shape. Draw the missing lines on the sketch to complete the perspective view.
The following are unfoldings of regular polyhedra.

1. Draw unfoldings of the above five different regular polyhedra, so that all have the same edge lengths. Fold these to assemble the polyhedra.

2. Opposite faces of a die always have a total of seven spots. Remembering this, work out how to cut out from the drawing on the right to make an unfolding of a die.

Where do we need to add pasting tabs?
### Further Topics

- The World of Finite Number Systems (Development) 174
- Representing Inequalities (Development) 178
- Incircle and Circumcircle of a Triangle (Development) 180
- Creating Problems to be Solved with Equations 184

### Let's Research

- Calculating Pi 186
- Adventures with Cubes 188
- Steps that are Kind to the User 192

### Supplementary Problems

Zu Chongzhi (see p. 187)
The World of Finite Number Systems

In elementary school you learned that $3 - 5$ "can't be done, but then in junior high school, you learned that $3 - 5 = -2$. But in everyday life there is a situation in which $3 - 5 = 10$. When would this be?

The conversation on the right is very natural. Expressed as an equation, it is:

$$10 + 5 = 3$$

**Problem 1**

Think of numbers as the numbers used on a clock face, and carry out the following calculations.

1. $5 + 7$
2. $5 + 12$
3. $5 + 10$
4. $10 + 10$
5. $7 - 5$
6. $5 - 12$
7. $5 - 10$
8. $10 - 10$

The system of numbers on a clock face, in which there are just 12 numbers, from 1 to 12, is called $\mathbb{Z}_{12}$.

**Problem 2**

The table on the following page shows the results of adding numbers in $\mathbb{Z}_{12}$. Fill in the blanks, to complete the table.

**Problem 3**

In the addition table for $\mathbb{Z}_{12}$, there is a number which plays the same role as 0 in ordinary addition. Which number is that?

In normal addition, zero has the following properties:

$$a + 0 = a, \quad 0 + a = a$$
In the above addition table, find the values of $x$ to satisfy the following:

1. $8 + x = 10$
2. $x + 7 = 12$
3. $12 + x = 7$
4. $9 + x = 5$
5. $x + 10 = 10$
6. $12 + x = 12$

If we know $a$ and $b$, we can find the value of $x$ to satisfy $a + x = b$ or $x + a = b$ by subtraction. We write this as $b - a$.

**Problem 5** Within $\mathbb{Z}_{12}$, calculate the following.

1. $10 - 4$
2. $5 - 7$

In the system called $\mathbb{Z}_{12}$, we use natural numbers 1 to 12, and think of the numbers as though on a clock face.
On a calendar, what do the 1st, 15th, and 22nd of the month have in common?

On a calendar, the dates are assigned to seven different days of the week. Since the 1st, 8th, 15th, 22nd, and 29th are all the same day of the week, we can regard them all as 1, then similarly regard all of the 2nd, 9th, 16th, 23rd, and 30th as 2. In this way we can divide the natural numbers into seven classes, to construct the system known as \( \mathbb{Z}_7 \). Another way of looking at this is to say that in \( \mathbb{Z}_7 \), numbers which give the same remainder when divided by seven are regarded as the same number.

If we regard 0, 7, 14, ... as 0, then the numbers making up \( \mathbb{Z}_7 \) are \{0, 1, 2, 3, 4, 5, 6\}.

**Problem 6**
Following the example of \( \mathbb{Z}_{12} \), complete the addition table for \( \mathbb{Z}_7 \) on the right.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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**Problem 7**
Using the table in Problem 6, calculate the following in \( \mathbb{Z}_7 \).

<table>
<thead>
<tr>
<th>1</th>
<th>5 + 3</th>
<th>2</th>
<th>3 + 4</th>
<th>3</th>
<th>2 - 4</th>
<th>4</th>
<th>0 - 5</th>
</tr>
</thead>
</table>

**Problem 8**
In a certain year, January 1st is a Friday. In the same year, what day of the week is February 1st?
In the same way as $\mathbb{Z}_7$, let's construct $\mathbb{Z}_5$ with \{0, 1, 2, 3, 4\}.

**Problem 9** In the same way as $\mathbb{Z}_7$, complete the addition table on the right for $\mathbb{Z}_5$.

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</tbody>
</table>

Is it also possible to do multiplication in $\mathbb{Z}_5$? For example, let's consider $2 \times 3$ to be calculated as follows:

\[
2 \times 3 = 2 + 2 + 2 = 4 + 2 = 1
\]

**Problem 10** For multiplication in $\mathbb{Z}_5$, carry out the calculations as above, and complete the multiplication table on the right.

<table>
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</table>

**Problem 11** To calculate $2 \div 3$, we need to find the number $x$ such that $3 \times x = 2$. Using the multiplication table of Problem 10, calculate the following in $\mathbb{Z}_5$.

1. $2 \div 3$
2. $3 \div 2$
3. $1 \div 4$

**Problem 12** Check that the following relation holds in $\mathbb{Z}_5$.

\[
2 \times (3 + 4) = 2 \times 3 + 2 \times 4
\]
Representing Inequalities

Miss Hayashi goes to the cake store, buys a number of cakes at 180 yen each, and pays with a 1000-yen bill. Since she received some change, she then bought one soft drink for 120 yen. How many cakes did she buy?

Angela approached this trigger question as follows.

Suppose she bought \(x\) cakes, then
\[180x + 120 = 1000\]
Solve this equation to get
\[180x = 880\]
Therefore \(x = \frac{44}{9}\)
Answer \(x = \frac{44}{9}\) cakes

Brian thought about the trigger question as in the bubble on the right. Find the answers other than the four that Brian thought of for the number of cakes.

In the trigger question, all that we need is that the total cost of the cakes and soft drink is not more than 1000 yen. So if we suppose that Miss Hayashi bought \(x\) cakes, we can represent this by the following expression.

\[180x + 120 \leq 1000\]

\[\text{[1]}\]
An expression such as (1), which represents a number comparison, is called an inequality.

The inequality (1) holds when \( x \) is less than or equal to \( \frac{44}{9} \). We call the range of values for which an inequality holds the solution of the inequality. So the solution of inequality (1) is \( x \leq \frac{44}{9} \).

In the trigger question, since \( x \) is a natural number, the solution is: one, two, three, or four cakes.

We can obtain the solution \( x \leq \frac{44}{9} \) as follows.

\[
180x + 120 \leq 1000 \\
180x \leq 1000 - 120 \\
180x \leq 880 \\
x \leq \frac{44}{9} \quad \text{(2)}
\]

**Problem 2** Solve the inequality \( 2x - 5 > 1 \).

In the trigger question on the previous page, Celia constructed the inequality \( 1000 - 180x \geq 120 \), and began to solve it as follows.

\[
1000 - 180x \geq 120 \\
-180x \geq 120 - 1000 \\
-180x \geq -880
\]

This is how Celia continued, dividing both sides by \(-180\) to get:

\[
x \geq \frac{44}{9}
\]

which is different from the result (2).

When dividing both sides of an inequality by a negative number, what do you have to be careful of?

**Problem 4** Solve the inequality \(-5x - 2 < 18\).
In Example 2 and Problem 3 on page 136, you constructed the circle center $P$ that touches line segment $AB$ and two half lines $AC$ and $BD$, as shown in the diagram on the right.

Considering $AB$, $AC$, and $BD$ as (whole) lines, let's see if there are other circles that touch all three.

On a piece of paper, draw a diagram as above, then extend lines $CA$ and $DB$, and fold the paper so that these two lines exactly meet each other. What line is the line of the fold?

First let's consider circles that touch both of the lines $AC$ and $BD$.

The diagram on the right shows the diagram of the trigger question, with $CA$ and $DB$ extended, to intersect at point $E$. In this diagram, the circle with center $P$ is moved, shrinking its radius, while keeping it touching the lines $EC$ and $ED$. We can see that there will be a position at which the circle is within $\triangle AEB$, and just touches the side $AB$.

**Problem 1** What is the path of the center points of circles touching the two lines $EC$ and $ED$?
Let's think about constructing a circle touching all three sides of a triangle.

**Problem 2**

If I is the center of a circle touching all three sides of \( \triangle ABC \), as shown in the diagram on the right, then what property does point I have?

If we drop a perpendicular from center I to each side of \( \triangle ABC \), all of these perpendiculars have the same length. Therefore, point I lies on the bisector of \( \angle ABC \), and also on the bisectors of \( \angle ACB \) and \( \angle BAC \).

The circle that touches all three sides of a triangle is called the incircle of the triangle.

**Problem 3**

Describe a method for constructing the incircle of a triangle. In your exercise book, draw a triangle, and construct the incircle of the triangle.

**Problem 4**

If we extend the sides of a triangle, how many circles are there that touch all three (extended) sides?
If we extend the sides of a triangle, in addition to the incircle there are three circles touching all three lines, shown as a, b, and c in the diagram on the right. The centers of these three circles lie on the bisectors of the internal and external angles of the triangle.

We can look at the circle with center P shown on page 180 as being the circle touching \( \triangle AEB \) formed by the lines AB, AC, and BD on the outside, as shown in the first diagram below.

Problem 5

In the diagrams below, \( \angle CAB \) and \( \angle DBA \) are gradually decreasing. Answer the following questions about these diagrams.

1. Check that the circle with center P becomes the incircle.
2. What happens to the incircle of \( \triangle AEB \) in the first diagram?
Is there a circle that passes through all three vertices of a triangle?

Suppose we were able to draw a circle passing through all three vertices of \( \triangle ABC \), as shown in the diagram on the right, then what property would the center \( O \) have?

To think of a way of constructing a circle passing through all three vertices of a triangle, first let's consider the circles passing through two vertices.

Problem 6 What is the path of the center points of circles passing through the two vertices \( B \) and \( C \) of \( \triangle ABC \)?

The circle that passes through all three vertices of a triangle is called the circumcircle of the triangle.

Problem 7 Look at the diagram on the right, and describe a method of constructing the circumcircle of a triangle.

Problem 8 Draw various triangles in your exercise book, and construct the circumcircles of these triangles.
Creating Problems to be Solved with Equations

Using information from the following problem, let's create a new problem.

Bert sets out from home to his school, which is 900 meters away. Four minutes later, his elder brother Alan sets out from home, to catch up with his younger brother. If Bert walks at 50 meters per minute, and Alan walks at 70 meters per minute, how many minutes does it take Alan after leaving home to catch up with his brother?

Solve the above problem.

Problem 2 Based on the above problem, invent a new problem. Say which parts of your problem are different from the original problem, and which parts are the same.

Based on the original problem, Alice created this new problem.

A road 2 km long runs around a pond. A friend and I decide that we will run in opposite directions around this pond. Three minutes after I start running, my friend starts from the same point, how many minutes after I start running will we meet?
Problem 3 Consider Alice's problem, and say which parts you think are different from the original problem, which parts are the same, and if there are any parts of the problem which are incomplete.

Try filling in the missing details to complete the problem.

Using Alice's problem, again make your own variant problem.

Belinda and Caroline created the following two problems:

Belinda's problem

I start reading a 300-page book at ten pages every day. Two days after I start reading, my little sister Lucy starts reading at twelve pages every day. How many days does it take from the day I start reading until we have both read the same number of pages?

Caroline's problem

There are two tanks P and Q: P contains 2 liters of water, and Q contains 4 liters. Now, we start adding water to tank P at 5 liters per minute, and to tank Q at 1 liter per minute. How many minutes will it take until there is three times as much water in tank P as in tank Q?

Problem 5 Consider Belinda and Caroline's problems, and say which parts you think are different from the problem on the previous page, and which parts are the same.

Problem 6 Try creating various new problems.

Problem 7 What do all of these problems have in common?
Calculating Pi

We use the Greek letter pi (pronounced “pie”) to represent a constant which is the ratio of the circumference of a circle to its diameter. This is taken from the first letter of the Greek word “perimetrōn” (“perimeter”). Let’s look at some of the ideas people have had in attempting to find the value of pi.

The circumference of a circle is longer than the perimeter of a polygon enclosed within the circle, and shorter than the perimeter of a polygon enclosing the circle. By constructing regular polygons with 96 sides, the ancient Greek mathematician Archimedes was able to show that the value of pi lies between $\frac{10}{71}$ and $3\frac{1}{7}$.

1. Convert $\frac{10}{71}$ and $3\frac{1}{7}$ to decimal form, and check that the value of pi, approximately 3.141, lies between them.

2. If you arrange two each of the odd digits from 1 to 5 to get 113355, then take the first three, 113, as numerator and last three, 355, as denominator, this gives $\frac{355}{113}$. Check that this is a good approximation to pi.
After this, many people made attempts to find a numerical value for pi, by various means. Ludolph van Ceulen, a German mathematician living in the Netherlands obtained a value using a regular polygon with $2^{32}$ sides, but the calculation took most of his life.

How large a number is $2^{62}$?
Try thinking of $2^{10}$ as 1000.

Principal investigators and their correct results

<table>
<thead>
<tr>
<th>Investigator</th>
<th>Country</th>
<th>Century</th>
<th>Digits</th>
</tr>
</thead>
<tbody>
<tr>
<td>Archimedes</td>
<td>Greece</td>
<td>3rd century</td>
<td>2</td>
</tr>
<tr>
<td>Zu Chongzhi</td>
<td>China</td>
<td>5th century</td>
<td>5</td>
</tr>
<tr>
<td>François Viète</td>
<td>France</td>
<td>16th century</td>
<td>10</td>
</tr>
<tr>
<td>Ludolph van Ceulen</td>
<td>Netherlands</td>
<td>17th century</td>
<td>35</td>
</tr>
<tr>
<td>Katahiro Takebe</td>
<td>Japan</td>
<td>18th century</td>
<td>40</td>
</tr>
</tbody>
</table>

The value has now been found, using a supercomputer, to more than 1240 billion decimal places.

Look for more stories about pi in books or on the Internet.

- Starting from the 1,041,032,609,981st decimal place is a sequence of twelve consecutive 1s.
- There are many mnemonics for pi, such as the following:
  "How I need a drink, alcoholic of course, after the heavy lectures involving quantum mechanics."
  The number of letters in each word gives the corresponding digit.
- March 14th is sometimes called "Pi day," after the American date notation 3-14. It also happens to be Albert Einstein's birthday.
Adventures with Cubes

If we take the cube shown on the right, and view it exactly from the front of face AEFB, it appears as a square. In this position, if we take the intersections P and Q of the diagonals of opposite faces ABCD and EFGH, and use the line through P and Q as an axis, how will the shape we see change as we rotate the cube?

1. Make a cube, and check that as you rotate the cube the appearance does actually change as shown in the following diagrams.

   a       b       c

   d       e

2. The appearance changes in the sequence from (a) to (e). As this happens, how do the lengths of the sides of the shape you see change? Describe this in words.

   In diagram (c), edge AE is lined up with edge CG. At this point, rectangle DHFB is the shape of a section through the cube perpendicular to face ABCD and including diagonal DB.

   So what is the length of DB?
We can construct rectangle DHBF in diagram (c) as shown on the right. Explain this construction, and carry it out yourself.

The rectangle we constructed in 3 is the same shape as both A and B series of standard paper sizes.

Fold a sheet of A4 paper as shown below, to check that the length of the long side of the sheet is equal to the diagonal of a square whose sides are the length of the short side of the sheet.

Creating a cube from rectangles

By intersecting two of the rectangles constructed in 3 above perpendicular to each other at their center lines, we can view the eight corners as the corners of a cube.

Use the following procedure to construct a cube.

1. Using stiff paper, make two rectangles of the same size as constructed in 3, and cut a slot in each one as shown on the right.
2. Take the two rectangles you made in step (1), and slot them into each other at right angles, and fix with adhesive tape.
3. Join the corners of the rectangles with thread.
In a single cube, how many different ways can you see the pair of rectangles you made in 5?

Diagonals of a cube

Let's see why the diagonals of a cube all intersect at a single point.

The four diagonals of a cube are also the diagonals of the rectangles forming the cube you made in 5. Let's check this.

The intersection of the two rectangles you made in 5 is the line segment as shown on the diagram on the right, joining midpoints I and J of sides AC and EG of rectangle AEGC. The diagonals we are considering in 7 pass through midpoint O of line segment IJ. Similarly, the diagonals of rectangle DHFB also pass through midpoint O of line segment IJ.

Therefore, we can see that the diagonals of the cube all intersect at the single point O.

We can also see that the diagonals of a cube all intersect at a single point for the following reasoning:

1. As shown in the diagram on the right, diagonals AG and DF are diagonals of rectangle AFGD, and their intersection is midpoint O of line segments AG and DT.
2. The diagonals BH and AG are diagonals of rectangle ABGH, and their intersection is midpoint O of line segments BH and AG.

3. Since both of the diagonals DF and BH intersect diagonal AG at its midpoint O, diagonals AG, DF, and BH must all intersect at the single point O.

4. Similarly, if we consider the diagonals of rectangle DEFC, diagonals DF and CE must intersect at point O.

5. This means that the diagonals of the cube, AG, DF, BH, and CE all intersect at the single point O.

We can view a cube as being made up from six square pyramids, each with a face of the cube as base and the point of intersection of the diagonals O as vertex.

8. Make six pyramids from modeling card, one with each face as base, and the intersection of the diagonals as vertex, arrange them as shown in the diagram on the right, and join them together with adhesive tape.

Folding the model above so that the vertices of the pyramids all meet at the center makes a cube.

Now if you fold the model inside-out so the vertices are on the outside, then the result is the same as attaching a pyramid to each face of the cube, with a hollow inside that is the same size as the cube.
Steps that are Kind to the User

The steps in the photograph below also include a ramp.

Let's look at the angles of the steps and ramp.

The following text is part of the Osaka Welfare City Planning Ordinance, laying down standards required for steps in public parks.

- The width must be at least 1.2 m.
- The riser height must be not more than 15 cm, and the tread width must be at least 35 cm.
- In a flight of steps, the riser height and tread width must be uniform.

The diagram on the right explains what is meant by the width of the steps, and by treads and risers.

Make a folded paper model of the steps shown above, with a width of 12 cm, a riser height of 1.5 cm, and a tread width of 3.5 cm.
For these steps, find the value in degrees of angle $x$ in the following diagram. Measure it using the model you made in 1.

Next, add a ramp to the model.

The following photograph shows a model of a flight of three steps, with a ramp included, and the diagram on the right represents a plan view of the steps and ramp.

Find the slope of the ramp above in degrees. Based on the lengths shown in the diagram above, draw a diagram of the ramp seen from the side, and measure the angle. Compare this with the angle you found in 2.

As you found in 3, putting the ramp at an angle makes the slope more gentle.

Let's look at the angles of the steps and ramp.
Calculation exercises with signed number

1. Calculate the following:
   1. \((-3) + (+12)\)
   2. \((+4) + (-7)\)
   3. \((+2) + (-5)\)
   4. \((-8) + (+8)\)
   5. \((-1) - (-9)\)
   6. \((+1) - (+6)\)
   7. \(0 - (-9)\)
   8. \((-3) - (+3)\)
   9. \(-2 + (-5) - (-6)\)
   10. \(8 - 13 + 7\)

2. Calculate the following:
   1. \((-7) \times (-6)\)
   2. \((-3) \times (-12)\)
   3. \((-18) \times \frac{5}{6}\)
   4. \(-6^2\)
   5. \((-5)^2\)
   6. \((-24) \div (-4)\)
   7. \((+64) \div (-8)\)
   8. \((-60) \div (-15)\)
   9. \(12 \div \left(-\frac{4}{3}\right)\)
   10. \((6) \times (-4) \times (-3)\)
   11. \(-16 \div 8 \times (-6)\)
   12. \(4 \times (-3)^2 \div (-12)\)

3. Calculate the following:
   1. \(15 + 3 \times (-2)\)
   2. \(18 \div (-3) - 4\)
   3. \(7 \times 4 - (-30) \div 2\)
   4. \(-6 \times (-5 + 8)\)

Let's try!

Fill in the blanks in the grids on the right, so that the totals for each row and column and for the two diagonals are all the same.

These are called "Magic squares".
### Calculation exercises in algebra

<table>
<thead>
<tr>
<th></th>
<th>Answer</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1</strong></td>
<td>Calculate the following:</td>
</tr>
<tr>
<td>1</td>
<td>4x + 2x</td>
</tr>
<tr>
<td>2</td>
<td>5a - 6a</td>
</tr>
<tr>
<td>3</td>
<td>3b + b + 6b</td>
</tr>
<tr>
<td>4</td>
<td>4y + 3 - 9y - 1</td>
</tr>
<tr>
<td>5</td>
<td>(2a + 3) + (3a + 1)</td>
</tr>
<tr>
<td>6</td>
<td>(7x - 6) + (x + 3)</td>
</tr>
<tr>
<td>7</td>
<td>(3x + 4) + (4x + 5)</td>
</tr>
<tr>
<td>8</td>
<td>(8a - 5) - (6a - 9)</td>
</tr>
<tr>
<td>9</td>
<td>(4a - 1) - (7a + 4)</td>
</tr>
<tr>
<td>10</td>
<td>(x - 7) - (3x + 1)</td>
</tr>
</tbody>
</table>

| **2** | Calculate the following: |
| 1 | 4/ \times 7 |
| 2 | 1/6 \times 12 |
| 3 | (-24y) \times 1/8 |
| 4 | (-5) \times (-9a) |
| 5 | 2(4x - 3) |
| 6 | (3x + 6) + (3) |

| **3** | Calculate the following: |
| 1 | 3(a + 2) + (2a - 3) |
| 2 | 5(-x + 1) + 4(2x + 3) |
| 3 | 2(3x - 4) - 3(2x - 3) |
| 4 | 2(y + 6) - 7(-2y - 1) |

| **4** | Find the values of the following expressions when a = -2. |
| 1 | 5a + 6 |
| 2 | 4 - 3a |
| 3 | 2a^2 |

**Let's try!**

I bought some soft drinks in cans costing $a$ yen each. Which of the following offers gives the lower price per can: “Buy five and get one free” or “Buy five, and we give 20% discount”? 

---

**Supplementary Problems** 195
Exercises in solving equations

1. Solve the following equations:
   1. \(x - 4 = -7\)
   2. \(4x = 32\)
   3. \(\frac{1}{3}x = -3\)
   4. \(x + 6 = 13\)
   5. \(-5x = -30\)
   6. \(\frac{4}{5}x = -16\)

2. Solve the following equations:
   1. \(4x - 3 = 5\)
   2. \(x = 7x + 6\)
   3. \(7 = -2x - 1\)
   4. \(11x - 15 = 8x\)
   5. \(8x + 29 = 3x - 11\)
   6. \(5x - 1 = 6x - 5\)
   7. \(17 - 2x = 4x - 7\)
   8. \(9x + 13 = x - 3\)
   9. \(5x + 4 = 28 - 3x\)
   10. \(10x - 19 = 13x + 8\)
   11. \(-4x + 9 = 9x - 16\)
   12. \(26 - 7x = 2x - 8\)

3. Solve the following equations:
   1. \(5x - 2(x - 3) = 9\)
   2. \(4(2x + 3) = x - 9\)

Let's try!

Around 1700 years ago, the Greek mathematician Diophantus was said to have the following text engraved on his tombstone. Can you work out what age he lived to?

"For a sixth of his life Diophantus was a child, after a further twelfth of his life he grew a beard. After another seventh of his life he married, and five years later a son was born. The son lived just one half of Diophantus's life, and died four years before his father."

Try drawing a diagram.
Solving problems using graphs

1. In the diagram on the right, (1) is the graph of \( y = ax \), and A is a point on this graph. (2) is the graph of \( y = \frac{b}{x} \), and P is a point where this graph intersects graph (1). If the coordinates of point A are (10, 15), and the \( x \) coordinate of P is 6, answer the following questions.

   1. Find the values of the constants of proportionality \( a \) and \( b \).

   2. How many points are there on graph (2) such that both \( x \) and \( y \) coordinates are integers?

2. A is a point on the graph of inverse proportionality \( y = \frac{6}{x} \). With B as point (5, 0), join origin O and points A and B to form triangle AOB. When the \( x \) coordinate of point A is positive, and the area of triangle AOB is 5 cm\(^2\), find the coordinates of point A. (Use gradations of centimeters.)

**Let's try!**

Alan and Bob race 100 meters. The first time, as Alan crosses the finishing line, Bob is still 4 m behind. The second time, Alan starts from 4 m behind the starting line. This time, will they reach the finishing line at the same time? Assume that both run at the same speed as the first time.
Find the following points and lines by construction.

1. In $\triangle ABC$ in the diagram on the left, point $P$ on side $AB$ such that $AP = CP$.

2. In $\triangle ABC$ in the diagram on the left, the line segment about which the triangle should be folded so that sides $BC$ and $BA$ coincide.

3. In the diagram on the left, the right-angled triangle $ABC$ such that $\angle BAC = 90$ degrees, and $AB = AC$.

Let's try!

A triangular patch of ground is surrounded by a fence, as shown in the diagram on the left. I want to start from point $P$, and run to touch the fence at some point on each of sides $AB$ and $BC$, then return to $P$. Draw the path I must take to run the shortest distance.
Exercises in finding surface area and volume

1. In the diagram on the right, the three circles have the same radius, and each circle passes through the centers of the other two circles. If each circle has a radius of 6 cm, find the total area of the colored parts.

2. The diagram on the right is an unfolding of a cone.
   1. Find the length of the slant height.
   2. Find the surface area.

3. Find the surface area and volume of a cylinder with a base radius of 4 cm and height of 9 cm.

4. Find the volume of a pyramid with a square base having a side of 3 cm and a height of 6 cm.

Let's try!

A container is in the shape of a prism with a square base, and has a capacity of 3 liters. Using only this container, find the simplest way to measure out 1 liter of water.
### Chapter 1 Positive and Negative Numbers

#### Chapter summary problems A

1. \( 5 > -9 \)
2. \( 4 < -1 \)
3. \( 5 < 2 \cdot 3 \)

2. \(-4, -3, -2, -1, 0, 1, 2, 3, 4\)

3. \(0, 2, 8\)
4. \(1, 7, 2, 6\)
5. \(0, 4, 4, 8\)
6. \(-1, 8, -7\)
7. \(1, 10, -6\)

5. \(1, 5, 2, 3, 2, 3, 4\)
6. \(-10, 4, 6\)
7. \(80, 6, 42\)
8. \(10, 8, 42\)

6. \(2, 1, 0\)
7. \(2, -1, 0, 1\)

#### Chapter summary problems B

1. \(\frac{1}{6} x + 1\)
2. \(\frac{1}{3} x + 2\)
3. \(200x\) yen

### Chapter 2 Algebra

#### Chapter summary problems A

1. When \(x = 3\)
   1. \(1\)
   2. \(15\)
   When \(x = -3\)
   1. \(-11\)
   2. \(21\)

2. \(5x, 2y\)
3. \(3x, 4 - \frac{3}{5}x\)
4. \(0, 2 - \frac{3}{x}\)
5. \(-4x + 1, 8 - 8\)

3. \(-12x + 42, 5x - 15\)
4. \(10x - 9, 3x - 4\)
5. \(7x - 36, 5x + 1\)
6. \(2x + 9, 6x - 22\)

4. \(\text{Sum } 8x + 8\) \(\text{Difference } 2x - 8\)
5. \(\text{Sum } -y - 2\) \(\text{Difference } -5y\)

#### Chapter summary problems B

1. \((16 + 4a)\) cm
2. \((x^2 - 17)\) pieces
3. \(a - b\) yen

1. \(\frac{1}{6} x + 1\)
2. \(\frac{1}{3} x + 2\)
3. \(200x\) yen

2. \(\left(x + \frac{2}{100}\right)\) students

4. The change from 1000 yen when buying three pencils and one notebook

5. \((8x + 4)\) go stones
Chapter 3 Equations

Chapter summary problems A  —  p. 82
1. \[ x = -12, \quad x = 2 \]
2. \[ x = 5, \quad x = -4 \]
3. \[ x = 6, \quad x = 1 \]
4. \[ x = -1, \quad x = 2 \]
5. \[ x = 3, \quad x = 0 \]
6. \[ x = 2, \quad x = -20 \]
7. \[ x = -1, \quad x = -1 \]

8 go stones

Cheaper notebooks are 90 yen each. I had 760 yen.

Chapter summary problems B  —  p. 83
1. \[ y = \frac{2}{3}x \]

2. \[ y = \frac{6}{x} \]

3. \[ Chapter summary problems B  —  p. 83
1. \[ y = 3x \]

2. \[ y = 48x \]

3. \[ S \text{ is proportional to } x, \]

4. \[ # \text{ is inversely proportional to } x. \]

Chapter 4 Proportionality and Inverse Proportionality

Chapter summary problems A  —  p. 111
1. \[ y = 18 - x, \quad y = -30 \]
2. \[ y = 80x \]
3. \[ y = \frac{3}{2}x \]
4. \[ y = -\frac{1}{x} \]
5. \[ y = -\frac{3}{x} \]

Chapter summary problems A  —  p. 112
1. \[ y = 3x \]

2. \[ y = 48x \]

3. \[ S \text{ is proportional to } x, \]

4. \[ # \text{ is inversely proportional to } x. \]

Chapter 5 Plane Figures

Chapter summary problems A  —  p. 138
1. \[ \text{Answer} 201 \]

2. \[ \text{Omitted} \]

3. \[ \text{Square, Regular hexagon, Circle} \]

Chapter summary problems A  —  p. 138
1. \[ \text{Answer} 201 \]

2. \[ \text{Omitted} \]

3. \[ \text{Line segment CB, Line segment CD} \]
Chapter 6 Three-Dimensional Figures

Chapter summary problems A — p. 168

1. Edges BE and CF
2. Edges AB, BC, and CA
3. Edges AD, BE, and CF
4. Faces BEFC, ABC, and DEF
5. Edges BC and EF

Chapter summary problems B — p. 139

1. Yes, true
2. Yes, true
3. Yes, true

Chapter summary problems B — p. 169

1. Yes, true
2. Yes, true

(3 cm)

Length of generator ..., 20 cm
Surface area ............... 224 \pi \text{ cm}^2
**Answers to Development Questions**

**Projections and Sections**  p.170~171

- **Cone**
  - 1. Rectangular parallelepiped
  - 2. Square pyramid
  - 3. Cylinder

- **Square**

<table>
<thead>
<tr>
<th>Number</th>
<th>Question</th>
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<tbody>
<tr>
<td>1</td>
<td>Omitted</td>
</tr>
<tr>
<td>2</td>
<td>Omitted</td>
</tr>
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</table>

**The World of Finite Number Systems**  p.174~177

**Representing Inequalities**  p.178~179

**Incircle and Circumcircle of a Triangle**  p.183

- **Monday**
  - 1. 1
  - 2. 0
  - 3. 5
  - 4. 2

- **Omitted**

- **Monday**
  - 1. Monday
  - 2. Tuesday
  - 3. Wednesday
  - 4. Thursday
  - 5. Friday
  - 6. Saturday
  - 7. Sunday

- **Monday**
  - 1. Omitted
  - 2. It becomes the circle below the triangle that touches one side and the extensions of the other two sides.
  - 3. The perpendicular bisector of side BC
  - 4. Construct the perpendicular bisectors of two sides of the triangle, then with their point of intersection as center, draw a circle whose radius is the distance to any of the vertices.
  - 5. Omitted

**Answer**  203
Answers to Supplementary Problems

Calculation exercises with signed numbers

1. | x | 3 | 2 | 1 | 1
   | y | 9 | 8 | 7 | 0
   | 4 | 5 | 6 | 7 |

2. | x | 12 | 11 | 10 | 9 |
   | y | 8 | 7 | 6 | 5 |

3. | x | 1 | 2 | 3 | 4 |
   | y | -1 | -2 | -3 | -4 |

Solving problems using graphs

1. \[ a = \frac{3}{2}, \ b = 5 \]

2. \[ \Lambda(3, 2) \]

Drawing construction exercises

Exercises in finding surface area and volume

1. \[ 30\pi \text{ cm}^2 \]

2. \[ 16 \text{ cm}, 132\pi \text{ cm}^3 \]

3. Surface area: \[ 104\pi \text{ cm}^2 \]
   Volume: \[ 144\pi \text{ cm}^3 \]

4. \[ 18 \text{ cm}^3 \]
### Index

**[a-o]**

- Rearranging terms 72
  \[9x - 5 = 2x + 23\]
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About Wasan

During the Edo period, Japanese mathematicians made their own independent advances, and this mathematical work is referred to as “wasan”. Look out for things to do with “wasan” around you.

Try the local museum

The museum in Ichinoseki City, Iwate Prefecture, has some writings on display relating particularly to the work of local “wasan” mathematicians.

Sanpo Doji Utaguruma (Old children’s poem for learning arithmetic)

Sangaku

Japanese mathematicians had their own way of publishing their work: it was engraved on a wooden tablet and framed, then the resulting item, known as “sangaku” was presented to a Buddhist temple or Shinto shrine. The tablet on the right is held in Furuo Hachiman Shrine in Kawagoe City, Saitama Prefecture.

Calculating rods and calculating board

The calculating rods are called “sangi”, and were introduced from China in the sixth or seventh century.
Making a cube and a regular tetrahedron with origami

1. How to fold a cube

2. Lift up the flap and tuck the mountain fold under.

3. Similarly, lift up the flap and tuck the mountain fold under.

4. How to fold a regular tetrahedron

5. When making the second fold, align the red dots (lower left corner to central fold).

6. Bring the red dots up to the vertex.
Put the two components together to form a cube.

Completed cube

Fix with adhesive tape.

Completed component of the cube - make two of these.

Make a central fold, then unfold the paper.

Completed regular tetrahedron

The tetrahedron just fits inside the cube!

Investigate the relation between the cube and tetrahedron you have made.
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