MATHEMATICS 2
Contents

1. Calculating with Formulas ........................................ 4
   1. Calculating with Formulas ................................ 4
   2. Using Algebraic Expressions .............................. 17

2. Simultaneous Equations ......................................... 24
   1. Simultaneous Equations ................................ 24
   2. Using Simultaneous Equations ......................... 36

3. Linear Functions ................................................ 46
   1. Linear Functions ........................................ 46
   2. Linear Functions and Equations ...................... 65

4. Parallel Lines and Congruent Figures ....................... 80
   1. Parallel Lines and Angles ............................. 80
   2. Congruent Figures .................................... 94

Alps and mountain tram (Switzerland)
Properties of Geometrical Figures

1. Triangles .......................... 110
2. Parallelograms ...................... 124
3. Triangles and Circles .............. 138

Probability ................................ 152

- Playtime problems ........................ 45, 79, 151
  Sassa-tale .............................. 45
  Equal Shares .......................... 79
  Rearrangement Puzzles .............. 151

- Appendix .................................. 169
  Further Topics .......................... 170
  Independent Research ................. 186
  Supplementary Problems .............. 194
  Answer .................................. 200
  Index .................................... 206

Hitachi Kamine leisure land
(Ibaraki prefecture)
Using this book

Contents of the Chapters

Opening pages to sections
These give an overview of the content covered by the section.
- Trigger question
- An exercise for thinking about a problem
- A problem connected to the main text

Main text

- Trigger question
- Practical example for understanding the content
  This question is a clue to solving the problem.
- A problem to check if you have correctly understood the basic content
- A problem for a better understanding of the content
- A problem for which a calculator is useful
- * Further description relating to the term in a footnote

Let's try! Extra problems to help you think about the content

At the end of each chapter...

Write in your exercise books:

- The useful techniques you have learned
- What you found interesting
- What you learned and enjoyed doing
Items below marked with an asterisk are further topics for you to pursue at your own pace.

<table>
<thead>
<tr>
<th>Problems</th>
<th>Basic problems</th>
<th>These cover subject matter you need to master. This shows where a topic is covered in the main text.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Chapter summary problems*</td>
<td>These problems review the whole chapter and show applications of what you have learned. These are grouped by difficulty into A and B* problems.</td>
</tr>
<tr>
<td></td>
<td>Let’s Investigate!*</td>
<td>These sections explore topics relating to what you studied in the chapter, in further depth.</td>
</tr>
<tr>
<td></td>
<td>Window on math*</td>
<td>These sections present topics of interest connected to what you are studying.</td>
</tr>
</tbody>
</table>

**Diversions**

These are puzzles which the math you have learned will help you to solve.

**Appendix**

Further Topics

These look at further developments of material you have covered.

Let’s Research

These show connections between mathematics and everyday life, in a way you can investigate yourself.

Supplementary Problems

These are review problems, testing your understanding of the subject matter.

The following are further development topics.

- Multiplying and Dividing Powers of $a$ ........ 14
- Simultaneous Equations of the Form $A=B=C$ ........ 44
- Graphs in step form ........................................ 49
- Graph of $y=k$ ............................................. 78
- Conditions for a rectangle, rhombus, and square ....................... 134
- Geometrical Figures Maintaining a Constant Angle ....... 147
- Probability of an Event Not Occurring ...................... 168
- Equations with Three Unknowns ......................... 170
- Finding the Number of Cases ......................... 74
- Making Predictions from Data ..................... 178

The sections labeled “Development” are not part of the formal curriculum requirements, but are provided to widen the scope of study.
Chapter 1 Calculating with Formulas

Let's play a number guessing game!

Think of a number.

1. Write the number you first thought of in ①.
2. Write 8 in ②.
3. Write the sum of ① and ② in ③.
4. Write the quotient of ③ divided by 2 in ④.
5. Write the difference when ② is subtracted from ④ in ⑤.
6. Write the product of ⑤ multiplied by 2 in ⑥.
7. Write the difference when ⑥ is subtracted from ③ in ⑦.

So the answer in ⑦ is 16, isn't it?
Try doing the same calculation using different numbers to start with.

<p>| | | | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>8</td>
<td>8</td>
<td>8</td>
<td>8</td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>(1 + 3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>(8 ÷ 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(4 - 3)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>(3 × 2)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>(3 - 3)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Why is the answer always the same? Write \( x \) for the number you start with, and calculate the values in 3 to 7 as expressions in \( x \), and try to see why.

If you change the number written in 2, how does the final result change?

Is the result in 7 always the same, just as when you wrote 8?
Monomials and Polynomials

Let's look at the number of terms and the number of variables in algebraic expressions.

Expressions consisting only of a product of letters and numerals such as $2x$ and $\frac{1}{3}a^2$ are called monomials. We also think of a single letter or numeral, such as $x$ or $-5$, as a monomial.

An expression such as $2x + 5$, $3a^2 + 4ab + 1$ which is the sum of a number of monomials, is called a polynomial. Each of the monomials in the polynomial is called a term.

Example 1) Since we can write $3x^2 - 2x - 5$ as:

$$3x^2 + (-2x) + (-5)$$

it is a polynomial, with the following terms:

$$3x^2, \quad -2x, \quad \text{and} \quad -5$$

Check 1) Name the terms of the polynomial $2x^2 - 4x + 3$.

Problem 1) Name the terms of the following polynomials.

1. $4a + 3b$
2. $2x + y - 3$
3. $\frac{1}{2}x - y^2 - \frac{1}{3}$
4. $mn + 3m^2n$
Degree of an Expression

In the following monomials, how many letters are multiplied?

1. \(3ab\)
2. \(-4x^2y\)

The number of letters multiplied in a monomial is called the **degree** of the monomial.

For example:

- **Two letters**
  - The degree of \(3ab\) is 2
    
    \[3ab = 3 \times a \times b\]

- **Three letters**
  - The degree of \(-4x^2y\) is 3
    
    \[-4x^2y = -4 \times x \times x \times y\]

**Check 2**

Give the degrees of the following monomials

1. \(-3a^2\)
2. \(x\)
3. \(\frac{1}{2}x^2y^3\)

The degree of a polynomial is the degree of the term with the highest degree. A polynomial of degree one is called **linear** (degree one). A polynomial with a degree of two is called a quadratic (degree two). The sequence continues with a cubic (degree three), quartic (degree four), and quintic (degree five).

**Example 2**

The degree of the polynomial

\[x^3 + 4x^2 - 5x\]

is three, and therefore this is a cubic.

**Check 3**

What is the degree of the polynomial \(2x^2 - 3x + 5\)?

**Problem 1**

Give the degrees of the following polynomials.

1. \(-4x + y\)
2. \(-3y^2\)
3. \(a^2b - ab + 2a\)
4. \(-s^2t^3 + t^2\)

1 — Calculating with Formulas
Calculations with Polynomials

Collecting Similar Terms

Try the following calculations.

1. $2a + 3a$
2. $4x - 2x - 6$

For example, in $5x + 7y - 3x + 6y$,
- $5x$ and $-3x$ are similar terms.
- $7y$ and $6y$ are similar terms.

Terms with common letters are called similar terms.

As shown on the right, we can combine similar terms into a single term, using the distributive law.

Example 1

1. $5x + 7y - 3x + 6y$

Rearrange the order of terms

Combine similar terms

$2x + 13y$

2. $4x^2 + 2x - 5x + 6x^2$

$= 4x^2 + 6x^2 + 2x - 5x$

$= 10x^2 - 3x$

Check 1

Collect similar terms in the following expressions.

1. $4x + 8y + 2x - 3y$
2. $5x^2 + 2x - 3x^2 - 4x$

Problem 1

Collect similar terms in the following expressions.

1. $8a - 7b - 3a + 5b$
2. $x^2 - 5x - x - 3x^2$
3. $4ab - 2a - ab + 2a$
4. $\frac{3}{2}x + \frac{1}{2}y - 2x + \frac{2}{3}y$
Addition and Subtraction of Polynomials

To add polynomials, all we need to do is to add all of the terms. To do this, we start by collecting similar terms.

We subtract polynomials in the same way, but first change the sign of each term in the polynomial being taken away.

Example 2

Adding polynomials

\[(3x + 4y) + (2x - 5y)\]
\[= 3x + 4y + 2x - 5y\]
\[= 3x + 2x + 4y - 5y\]
\[= 5x - y\]

Subtracting polynomials

\[(3x + 4y) - (2x - 5y)\]
\[= 3x + 4y - 2x + 5y\]
\[= 3x - 2x + 4y + 5y\]
\[= 5x - y\]

Check 2

Calculate the following:

1. \((x + y) + (3x + 2y)\)

2. \((3x - 2y) - (x + 5y)\)

Problem 2

Calculate the following:

1. \((x - 4y) + (5x - 3y)\)

2. \((-5x - 9 - 3y) + (6 + 5x - 8y)\)

3. \((2a^2 - 3a + 4) - (a^2 + 5 - a)\)

4. \(a + 2b - 3\)

Problem 3

For these two polynomials, answer the following questions:

\[a + 4b, \quad 4a - 2b\]

1. Find the sum of the two polynomials.

2. Find the difference when the polynomial on the right is subtracted from the polynomial on the left.
Multiplication and Division of Polynomials by Numbers

Let's calculate $4(x + 2)$

To multiply a polynomial by a number, we use the distributive law.

**Example 3**

$$-5(3x - y + 2)$$

$$= -15x + 5y - 10$$

**Check 3** Calculate $-3(x + 3y - 4)$

**Problem 4** Calculate the following:

1. $3(a + 4b)$
2. $-4(-2x + 3y)$
3. $6\left(\frac{a}{3} - \frac{b}{2}\right)$
4. $(6x - 8y - 4) \times \left(-\frac{1}{2}\right)$

To divide a polynomial by a number, we can convert the calculation to a multiplication.

**Example 4**

$$\frac{6a - 9b}{3}$$

$$= \frac{6a - 9b}{3} \times \frac{1}{3}$$

$$= \frac{6a}{3} - \frac{9b}{3}$$

$$= 2a - 3b$$

**Check 4** Calculate $(12x - 20y) \div 4$

**Problem 5** Calculate the following:

1. $(-9a + 12b) \div 3$
2. $(15x^2 - 5x + 30) \div (-5)$

10 --- Calculating with Formulas
More Calculations

Example 5

\[ 4(2x - y) - 3(2x - 5y) \]

Remove the parentheses

\[ = 8x - 4y - 6x + 15y \]

Collect similar terms

\[ = 2x + 11y \]

Problem 6

Calculate the following:

1. \( 2(x + 4y) + 3(x - 5y) \)
2. \( 4(3a - 2b) + 6(-a + 3b) \)
3. \( 3(3x - y) - 5(2x + y) \)
4. \( 3(x^2 + 4x - 2) - 2(6x - 1) \)

Problem 7

Find the result of subtracting 4 times \( x + 3y \) from 3 times \( 2x - 4y \).

Example 6

Calculate...

\[ \frac{3x - y}{2} - \frac{x - 4y}{4} \]

Make a common denominator

\[ = \frac{2(3x - y) - x - 4y}{4} \]

Combine into a single fraction

\[ = \frac{2(3x - y) - (x - 4y)}{4} \]

Remove the parentheses

\[ = 6x - 2y - x + 4y \]

Collect similar terms

\[ = 5x + 2y \]

Problem 8

Calculate the following:

1. \( \frac{7x - 4y}{10} + \frac{x + 2y}{3} \)
2. \( \frac{5x - y}{3} + \frac{3x + y}{2} \)
3. \( \frac{2a + b}{3} - \frac{a - 2b}{6} \)
4. \( x + y - \frac{x - 6y}{3} \)


Multiplication and Division of Monomials

Multiplication

Consider the area of a rectangle of height $3a$ cm and width $4b$ cm: what multiple is this of the area of a rectangle of height $a$ cm and width $b$ cm?

To find the product of monomials, such as $3a \times 4b$, we can multiply the product of the coefficients by the product of the letters.

Example 1

\[8x \times (-4y) = 8 \times (-4) \times x \times y\]
\[= -32xy\]

Check 1

Calculate $3x \times (-6y)$

Problem 1

Calculate the following:

1. $5x \times 4y$
2. $(-3n) \times (-2m)$
3. $(-2ab) \times 4c$
4. $\frac{1}{3}y \times 6x$

Example 2

1. $2a \times 3a^2 = 6a^3$
2. $(-4m)^2 = 16m^2$

Check 2

Calculate the following:

1. $5a \times (-a^2)$
2. $(-2x)^2$

Problem 2

Calculate the following:

1. $ab \times 4ab^2$
2. $(-a)^3 \times 2b$
Division

A rectangle has width $4b$ cm and area $12ab$ cm$^2$: what is its height in cm?

We can carry out a division of monomials, $12ab \div 4b$ as shown on the right.

We can cancel letters in the numerator and denominator, just like numbers.

Example 3

1. \[8xy \div (-2x) = \frac{8xy}{-2x} = -\frac{8xy}{2x} = -\frac{4 \cdot 2 \times \frac{1}{2} \times \frac{3}{2}}{4 \times \frac{1}{2}} = -4y\]

2. \[\frac{1}{2}a^2b \div \frac{2}{3}a = \frac{a^2b}{2} \div \frac{2a}{3} = \frac{a^2b}{2} \times \frac{3}{2a} = \frac{3}{4}ab\]

Check 3

Calculate the following:

1. \[6ab \div 3a\]

2. \[(-10xy) \div \frac{1}{2}x\]

Problem 3

Calculate the following:

1. \[9xy \div (-3xy)\]

2. \[8x^2 \div (-6x)\]

3. \[(-4xy^2) \div \frac{1}{2}xy\]

4. \[\frac{2}{3}b^2c \div \frac{5}{6}bc^2\]
Calculations Combining Multiplication and Division

**Example 4**
\[ a^2 b \div b = \frac{a^2 b}{a b} = \frac{a \times a \times b}{a \times b} = \frac{a}{a} = b \]

**Check 4** Calculate \( a^2 \times b \div 3 a b \)

**Problem 4** Calculate the following:
1. \( a^2 b \div a b^2 \times 3 \)
2. \( 8 \times 3 \div (-4 \cdot x) \div x \)
3. \( (-2 \cdot x)^3 \times x \div (-2 \cdot x) \)

---

**Multiplying and Dividing Powers of a**

**Trigger**

What number goes \( \square \) in the following equations?
1. \( a^3 \times a^4 = a^{11} \)
2. \( a^5 \div a^2 = a^3 \)

In **Example 1**, \( a^3 \times a^4 = (a \times a \times a) \times (a \times a \times a \times a) = a^7 \) and we can find the index of \( a^7 \) from \( 3 + 4 \).

**Problem 1** Evaluating \( (a^2)^3 \) yields what power of \( a \)? And how can we find this index?

In **Example 2**, we can find the index 3 of the quotient \( a^3 \) from \( -2 \).

**Problem 2** Calculate \( a^4 \div a^4 \), and see if you can see what value we should consider \( a^0 \) to have.
Let's find the value of expressions including two or more letters.

**Example 1**

When \(x = 2\) and \(y = -4\), find the value of \(3x - 5y\).

\[
3x - 5y = 3 \times 2 - 5 \times (-4) = 6 + 20 = 26
\]

Answer 26

**Check 1**

When \(x = -3\) and \(y = 2\), find the values of the following expressions.

1. \(3x - 5y\)
2. \(-x + 3y^2\)

**Trigger**

When \(a = 5\) and \(b = -3\), let's find the value of the following expression.

\[
2(3a - 4b) - 4(a - 3b)
\]

We can evaluate expression (1) by substituting \(a = 5\) and \(b = -3\), to obtain:

\[
2 \times [3 \times 5 - 4 \times (-3)] - 4 \times [5 - 3 \times (-3)]
\]

But let's see if there is a simpler way.

When evaluating an expression, before making the substitutions it may help to simplify first.

**Problem 1**

Answer the following problems about the expression (1).

1. Simplify the expression (1).
2. Substitute \(a = 5\) and \(b = -3\) into the expression you obtained in (1), to evaluate the expression.

**Problem 2**

When \(a = -2\) and \(b = \frac{1}{3}\), evaluate the following expressions.

1. \(4(a + 2b) + (a - 5b)\)
2. \(8a^2b \div 4a\)
Basic Exercises

1. Terms and degrees of polynomials
   For the polynomial $2x^2 - 5x + 9$, answer the following questions.
   ① Name the terms.
   ② What is the degree of this polynomial?

2. Addition and subtraction of polynomials
   Calculate the following:
   ① $2a - 3b + 4a + 7b$
   ② $3x^2 - 4x - 2x^2 + 6x$
   ③ $(2a + 3b) + (a - 6b)$
   ④ $(4x + y) - (3x - 5y)$
   ⑤ $(-2a + 5b) - (-2a + 7b)$

3. Calculating with linear polynomials and numbers
   Calculate the following:
   ① $-3(2x + y)$
   ② $(28a - 4b) ÷ 4$
   ③ $2(a + b) + 5(2a - b)$
   ④ $3(x - 2y) - 2(2x - 5y)$

4. Multiplication and division of monomials
   Calculate the following:
   ① $(-4a) × 5b$
   ② $3pq^2 × 2p$
   ③ $(-3a)^2$
   ④ $5ab ÷ \frac{5}{6} a$
   ⑤ $3x^2y ÷ 6xy$
   ⑥ $ab^2 ÷ b × 4a$

5. Evaluating expressions
   When $a = 2$ and $b = -3$, find the value of $4a - 5b$. 

16 1 — Calculating with Formulas
Consider the sum of five consecutive integers:

- What properties does this sum have?
- Try investigating some examples.

Let's try to represent the properties we investigated above by using algebra.

In a pair of consecutive integers, such as 5 and 6, the second number is one more than the first number. Thus if the first number is \( n \), the following number is \( n + 1 \).

**Example 1**
The sum of five consecutive integers is a multiple of 5.

Explain this, using algebra.

**Answer**
Of the five consecutive integers, let the smallest be \( n \). Then the five consecutive integers are:

\[
\begin{align*}
&n, \quad n + 1, \quad n + 2, \quad n + 3, \quad n + 4
\end{align*}
\]

Their sum is therefore:

\[
\begin{align*}
n + (n + 1) + (n + 2) + (n + 3) + (n + 4) &= 5n + 10 \\
&= 5(n + 2)
\end{align*}
\]

Since \( n + 2 \) is an integer, \( 5(n + 2) \) is a multiple of 5.

Therefore the sum of five consecutive integers is a multiple of 5.

**Problem 1**
In Example 1, make the same explanation, but setting \( n \) as the value of the middle number.
Trigger
Consider the integer of the same numeral where ones and ten places are reversed.
Check with different examples that the sum of these two integers is always a multiple of 11.

Let's now show why this property holds for any two-digit integer, using algebra.

If the tens digit is \( x \) and the units digit is \( y \), we can represent any two-digit integer as \( 10x + y \).

The sum of a two-digit integer and the same integer with the units and tens digits reversed is always a multiple of 11.

Show this, by using algebra.

Answer
Suppose the tens digit of the original integer is \( x \) and the units digit is \( y \), then:
- The first number is \( 10x + y \)
- The reversed number is \( 10y + x \)

Thus the sum is:
\[
(10x + y) + (10y + x) = 11x + 11y
\]
\[
= 11(x + y)
\]

Since \( x + y \) is an integer, \( 11(x + y) \) is a multiple of 11.

Therefore, the sum of a two-digit integer and the same integer with the units and tens digits reversed is always a multiple of 11.

Problem 2
For the pair of integers we considered in the Trigger, consider their difference. What can you say about it?

Explain this, using algebra.
Using algebraic calculations, let's look at some graphical problems.

**Example 3**

If the arc length of a sector is \( \ell \), and the radius is \( r \), the area \( S \) is given by:

\[
S = \frac{1}{2} \ell r
\]

Prove this.

**Answer**

If the central angle of the sector is \( \alpha \) degrees, then:

\[
\ell = 2\pi r \times \frac{\alpha}{360} \quad (1)
\]

\[
S = \pi r^2 \times \frac{\alpha}{360} \quad (2)
\]

If we divide both sides of equation (1) by \( 2\pi r \), we can express the \( \frac{\alpha}{360} \) that is common to both equations, in terms of \( \ell \) and \( r \) thus:

\[
\frac{\alpha}{360} = \frac{\ell}{2\pi r}
\]

Substituting this into (2) gives:

\[
S = \pi r^2 \times \frac{\ell}{2\pi r} = \frac{1}{2} \ell r
\]

Therefore the area \( S \) of the sector is given by:

\[
S = \frac{1}{2} \ell r
\]

*This looks rather like the formula for the area of a triangle.*

**Problem 3**

A sector has a radius of 5 cm and an arc length of \( 2\pi \) cm.

Find the area of this sector.

**Problem 4**

A circle has a radius of \( r \). If we double the radius of this circle, to what multiple does the area increase? If we halve the radius, what happens to the area?
Transforming Equations

To buy 30 tennis balls, I decide to buy a combination of cans holding two balls and cans holding three balls.

What combinations can I buy?

(1) If the number of cans holding two balls is \( x \), and the number of cans holding three balls is \( y \), write the relation between \( x \) and \( y \).

(2) If I buy six of the cans holding two balls, how many three-ball cans must I buy?

When considering cases such as the above, it is helpful to make an equation giving \( y \) in terms of \( x \).

Example 1: From the following equation (1), derive an expression for \( y \) in terms of \( x \).

\[ 2x + 3y = 30 \] .......................... (1)

Move \( 2x \) to the other side:

\[ 3y = -2x + 30 \]

Divide both sides by 3:

\[ y = \frac{-2}{3}x + 10 \] .......................... (2)

Transforming the equation (1) relating \( x \) and \( y \) into an equation like (2) giving \( y \) in terms of \( x \) is called solving equation (1) for \( y \).

Problem 1: Using the equation we derived in Example 1, find all combinations of both can sizes with which we can buy 30 tennis balls.

Check 1: Solve the following equations for the letters in brackets.

(1) \( x + 2y = 5 \) \( \{ x \} \)  
(2) \( 2xy = 4 \) \( \{ y \} \)
Problem 2: Solve the following equations for the letters in brackets.

1. \( \ell = 2(a + h) \)  \( [ \ell ] \)  
2. \( V = \frac{1}{3} a^2 h \)  \( [ h ] \)

Basic Exercises

1. Using algebra for explanation
   For any two-digit integer, subtracting the sum of the digits from the number gives a multiple of 9. Explain this using algebra.

   Cylinder A has a base radius of \( r \) cm and a height of \( h \) cm, while cylinder B has a base radius twice that of A and height one-half that of A. What is the ratio of the volumes of cylinders A and B?

2. Solve the following equations for the letters in brackets.

   1. \( 3x - 2y = 4 \)  \( [ y ] \)
   2. \( \ell = 2\pi r \)  \( [ r ] \)
Chapter Summary Problems A

1 Calculate the following:
   (1) \(4a - 3b + 5b - 6a\)
   (2) \(7x + 2y - 4x - 3y\)
   (3) \((4x - 7y) + (3x - 5y)\)
   (4) \((5x^2 - 4x) - (x^2 - 4x)\)

2 Calculate the following:
   (1) \(3(2a - 3b)\)
   (2) \((a + 4b) \times (-2)\)
   (3) \((2a - 6b) \div 2\)
   (4) \((10x^2 - 15x) \div (-5)\)
   (5) \(3(2a + b) + 4(a - 2b)\)
   (6) \(2(x^2 + 6x) - 3(4x - 1)\)

3 Calculate the following:
   (1) \(6x \times (-3x)\)
   (2) \((-a)^2 \times 4a\)
   (3) \(4ab \div (-8b)\)
   (4) \(9x^2 \div (-x)\)
   (5) \(5x^2y \div \frac{x}{3}\)
   (6) \(a^2 \times 8b \div 4ab\)

4 When \(x = 3\) and \(y = -\frac{1}{3}\), find the values of the following expressions.
   (1) \((x + 2y) - (3x - 4y)\)
   (2) \(24xy^2 \div (-6y)\)

5 Solve the following equations for the letters in brackets.
   (1) \(3x - 4y + 2 = 0\) \(\{y\}\)
   (2) \(m = \frac{a + b}{2}\) \(\{a\}\)

6 The sum of 2, 4, and 6 is 12, which is a multiple of 6. In fact, the sum of any three consecutive even number is a multiple of 6. Show that this is true for any even numbers using algebra. A multiple of 2 is called an even number.
Chapter Summary Problems B

1. Calculate the following:
   (1) \( \frac{2x + y}{2} + \frac{x - y}{3} \)
   (2) \( x^2 + x - \{3x - (x^2 + 1) + 5\} \)

2. If \( A = x + y \) and \( B = 2x - 3y \), calculate the values of the following expressions.
   (1) \( 4A - 3B \)
   (2) \( A \cdot (B - 2A) \)

3. In the diagram on the right, find the area of the shaded part.

4. In the monthly calendar on the right, the sum of the three vertically adjacent numbers is equal to three times the central number. This holds for any three such numbers. Explain this using algebra.

5. In the number guessing game on pages 4 and 5, if we change the number in step 2, what should we do to get the final answer? Try making \( x \) the first number thought of, and \( y \) the number in step 2.

Let's investigate!

For the arrangement of numbers in a monthly calendar, see if you can find other relationships like that in Problem 4 above, which always hold. Explain any rule you find, using algebra.
Congratulations on winning!

Girls basketball team

Thanks to the efforts of all members, the girls basketball team achieved first place in the area tournament for the fourth year running. The captain, Hayashi, played a big role with a series of shots, scoring 24 points, excluding free throws.

<table>
<thead>
<tr>
<th>Player</th>
<th>Points</th>
<th>Number of successful shots</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hayashi</td>
<td>24</td>
<td>10</td>
</tr>
<tr>
<td>Ishikawa</td>
<td>15</td>
<td>6</td>
</tr>
<tr>
<td>Yamaguchi</td>
<td>11</td>
<td>5</td>
</tr>
</tbody>
</table>
In basketball, aside from free throws, there are three-point shots, and two-point shots. How many successful shots, two- and three-point did Captain Hayashi make?

Let \( x \) be the number of Hayashi's successful three-point shots, and \( y \) be the number of successful two-point shots. Then for the total of 24 points, write an equation showing the relation between \( x \) and \( y \).

From the equation you wrote above, find all possible combinations of shots.

<table>
<thead>
<tr>
<th>Number of three-point shots</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of two-point shots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

And what else do we need to know to decide the number of shots?
Simultaneous Equations and Their Solutions

Hayashi says:

"I scored a total of ten shots, including both three-point and two-point shots."

Let's add this condition, and look at the next question on the previous page.

The above condition can be written as follows:

\[ x + y = 10 \]  \hspace{1cm} (1)

An equation such as the one you created on the previous page, or (1) above is called a linear equation in two unknowns.

Problem 1) Find the combinations of \( x \) and \( y \) values that satisfy equation (1), and enter them in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem 2) Using the table on the previous page and the table in Problem 1, find the common pairs of \( x \) and \( y \) values.

Consider:

\[
\begin{align*}
3x + 2y &= 24 \\
2x + y &= 10
\end{align*}
\]

Two or more equations such as this are called simultaneous equations. A combination of values for the letters such that all of the equations hold is called a solution. Finding the solution is called solving the simultaneous equations.

The solution to the simultaneous equation in problem 2 is \( x = 4 \) and \( y = 6 \).

Check 1) Which of the following combinations of values for \( x \) and \( y \) are solutions to the simultaneous equations:

\[
\begin{align*}
2x + y &= 11 \\
x - 2y &= 3
\end{align*}
\]

(a) \( x = 6, \ y = -1 \)  \hspace{1cm} (b) \( x = 7, \ y = 2 \)  \hspace{1cm} (c) \( x = 5, \ y = 1 \)
Solving Simultaneous Equations

In a certain fruit store, the cost of two apples and five oranges is a total of 600 yen, while the cost of two apples and three oranges is 480 yen. The illustration below shows how to find the cost of a single orange using diagrams. Fill in the blue boxes with the correct numbers.

\[
\begin{align*}
\text{2 apples and 5 oranges} & \quad \text{yen} \\
\text{2 apples and 3 oranges} & \quad \text{yen}
\end{align*}
\]

Comparing above two lines
\[
\begin{align*}
\text{2 apples} & \quad \text{yen} \\
\text{1 apple} & \quad \text{yen}
\end{align*}
\]

Therefore
\[
\begin{align*}
\text{3 oranges} & \quad \text{yen} \\
\text{1 orange} & \quad \text{yen}
\end{align*}
\]

Writing the relations shown on the left in algebraic form using \(x\) and \(y\) gives the equations on the right. By transforming the simultaneous equations in this way you obtain an equation having one unknown. This becomes a linear equation with one unknown which you learned to solve in year 1.

\[
\begin{align*}
2x + 5y &= 600 \\
2x + 3y &= 480
\end{align*}
\]

Subtracting the lower equation from the upper equation gives
\[
2y = 120
\]

Therefore
\[
y = 60
\]

Method of Addition and Subtraction

Consider the following system of equations:
\[
\begin{align*}
2x + 5y &= 600 \\
2x + 3y &= 480
\end{align*}
\]

The coefficient of \(x\) is the same in both equations. Therefore if we take the difference of the left sides and the difference of the right sides, we eliminate the term in \(x\), leaving a linear equation in \(y\) only.

\[
\begin{align*}
A &= B \\
C &= D
\end{align*}
\]

Therefore
\[
A - C = B - D
\]
Given the simultaneous equations \[
\begin{align*}
2x + 5y &= 600 \\
2x + 3y &= 480
\end{align*}
\]
The following summarizes how to solve them.

### Answer

\[
\begin{align*}
2x + 5y &= 600 \\
2x + 3y &= 480
\end{align*}
\]
Subtract each side of (2) from the corresponding side of (1) to give:

\[
\begin{align*}
2x + 5y &= 600 \\
- \quad 2x + 3y &= 480
\end{align*}
\]
\[
2y = 120
\]
\[
y = 60
\]

Substitute (3) into (1) to find the value of \(x\):

\[
2x + 5 \times 60 = 600
\]
\[
x = 150
\]

Answer: \(x = 150, y = 60\)

Substitute these values of \(x\) and \(y\) into the original simultaneous equations to check.

In (1):

\[
\text{LHS} = 2 \times 150 + 5 \times 60 = 600 \quad \text{RHS} = 600
\]

In (2):

\[
\text{LHS} = 2 \times 150 + 3 \times 60 = 480 \quad \text{RHS} = 480
\]

**Problem 1**

As a way of solving the simultaneous equations above, substitute (3) into (2) to find the value of \(x\), and compare the result with the above solution.

On line 6 of the answer, the equation \(2y = 120\) does not include \(x\), and is an equation in the single unknown \(y\). The variable \(x\) was eliminated using the two given equations.

28 — Simultaneous Equations
Consider a pair of simultaneous equations where the coefficients of one variable are opposite in sign but with equal absolute values. Let's look for a way how to eliminate this variable.

**Example 1**

Solve the following simultaneous equations.

\[
\begin{align*}
-2x + 3y &= 4 \\
2x + 5y &= 12
\end{align*}
\]

**Hint**

In the above simultaneous equations, if we add them together we can eliminate one letter.

\[
\begin{align*}
A &= B \\
C &= D
\end{align*}
\]

\[
A + C = B + D
\]

**Answer**

Add each side of (1) to the corresponding side of (2) to give:

\[
\begin{align*}
-2x + 3y &= 4 \\
+ & \quad 2x + 5y = 12 \\
\hline
8y &= 16 \\
y &= 2
\end{align*}
\]

Now substitute (3) into (2) to find the value of \(x\):

\[
2x + 5 \times 2 = 12 \\
2x = 2 \\
x = 1
\]

Answer: \(x = 1, \ y = 2\)

**Check 1**

Solve the following simultaneous equations.

\[
\begin{align*}
(1) \quad & \begin{align*}
3x + 2y &= 18 \\
x + 2y &= 14
\end{align*} \\
(2) \quad & \begin{align*}
-4x + 7y &= 2 \\
4x - 9y &= 2
\end{align*}
\]

I wonder if we should add the equations? Or subtract them?

**Problem 2**

Solve the following simultaneous equations.

\[
\begin{align*}
(1) \quad & \begin{align*}
x + y &= -3 \\
x - y &= 7
\end{align*} \\
(2) \quad & \begin{align*}
2x + y &= 2 \\
-y + 5x &= -9
\end{align*}
\]

1 — Simultaneous Equations 29
In a fruit store, the total cost of three apples and one orange is 350 yen, while the cost of four apples and three oranges is 550 yen. Find the cost of a single apple and a single orange.

If the absolute values of the coefficients are different, how can we eliminate one variable?

Example 2 Solve the following simultaneous equations.

\[
\begin{align*}
3x + y &= 350 \quad \text{(1)} \\
4x + 3y &= 550 \quad \text{(2)}
\end{align*}
\]

Answer

\[
\begin{align*}
(1) \times 3 & \quad 9x + 3y = 1050 \\
(2) & \quad 4x + 3y = 550 \\
\hline
5x & = 500 \\
x & = 100
\end{align*}
\]

(3) into (1) to find the value of \( y \).

\[
3 \times 100 + y = 350
\]

\[
y = 50
\]

Answer \( x = 100, \ y = 50 \)

Check 2 Solve the following simultaneous equations.

\[
\begin{align*}
\text{(1)} & \quad 4x + 3y = 1 \\
x + 2y & = 4 \\
\text{(2)} & \quad 2x - y = 4 \\
5x + 3y & = -1
\end{align*}
\]

Problem 3 Solve the following simultaneous equations.

\[
\begin{align*}
\text{(1)} & \quad 6x - 7y = 12 \\
-3x + 2y & = 3 \\
\text{(2)} & \quad 2x - 3y = -4 \\
4x + 5y & = -8
\end{align*}
\]
Example 3 Solve the following simultaneous equations.

\[
\begin{align*}
3x - 4y &= -15 \\
2x + 3y &= 7
\end{align*}
\]

Hint If we multiply each equation by an appropriate number, we can make the coefficients of one of \(x\) and \(y\) equal in absolute value.

Answer

\[
\begin{align*}
3x - 4y &= -15 \quad \text{(1)} \\
2x + 3y &= 7 \quad \text{(2)}
\end{align*}
\]

\(\text{(1)} \times 2 \quad 6x - 8y = -30 \quad \text{(3)}
\]

\(\text{(2)} \times 3 \quad 6x + 9y = 21\)

\[
\begin{align*}
-17y &= -51 \\
y &= 3 \quad \text{(3)}
\end{align*}
\]

Substitute (3) into (1) to give

\[
3x - 4 \times 3 = -15
\]

\[
3x = -3
\]

\[
x = -1
\]

Answer \(x = -1, \ y = 3\)

Problem 4 Solve the following simultaneous equations of Example 3 by eliminating \(y\).

Making the coefficients of one letter equal in absolute value, and adding or subtracting the left and right sides together to eliminate the letter, is called the method of addition and subtraction.

Check 3 Solve the following simultaneous equations.

\[
\begin{align*}
\begin{align*}
5x + 3y &= 2 \\
9x - 2y &= 11
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
4x + 7y &= -13 \\
5x + 2y &= 4
\end{align*}
\end{align*}
\]

Problem 5 Solve the following simultaneous equations.

\[
\begin{align*}
\begin{align*}
-3x + 7y &= -1 \\
5x - 4y &= 6
\end{align*}
\end{align*}
\]

\[
\begin{align*}
\begin{align*}
4x - 5y &= 21 \\
3x - 2y &= 21
\end{align*}
\end{align*}
\]
Substitution Method

In a fruit store, the total cost of two apples and five oranges is 650 yen, and we are told that the cost of an apple is ten yen more than the cost of two oranges. Find the cost of a single apple and a single orange.

We can use the following method to eliminate an unknown.

**Example 4** Solve the following simultaneous equations.

\[
\begin{align*}
2x + 5y &= 650 \quad \text{(1)} \\
x &= 2y + 10 \quad \text{(2)}
\end{align*}
\]

**Hint** Note that in (2) the expression 
\[2y + 10\] is equal to \(x\). We can substitute this expression for \(x\) in (1).

This way we can eliminate \(x\) from (1).

\[
\begin{align*}
2(2y + 10) + 5y &= 650 \\
x &= 2y + 10
\end{align*}
\]

**Answer**

\[
\begin{align*}
2x + 5y &= 650 \quad \text{(1)} \\
x &= 2y + 10 \quad \text{(2)}
\end{align*}
\]

Substitute (2) into (1) to give

\[
\begin{align*}
2(2y + 10) + 5y &= 650 \\
4y + 20 + 5y &= 650 \\
9y &= 430 \\
y &= 40 \quad \text{(3)}
\end{align*}
\]

Substitute (3) into (2) to give

\[
x = 2(40) + 10 = 150 \quad \text{Answer: } x = 150, \ y = 40
\]

This method of eliminating an unknown by substituting one equation into the other is called the substitution method.

32 2 — Simultaneous Equations
Check 4  Solve the following simultaneous equations, using the substitution method.

\[
\begin{align*}
(1) \quad \begin{cases} 
  y = 2x \\
  x + y = 6 
\end{cases} & \quad (2) \quad \begin{cases} 
  2x - 3y = -8 \\
  x = 4y + 1 
\end{cases}
\end{align*}
\]

Problem 6  Solve the following simultaneous equations, using the substitution method.

\[
\begin{align*}
(1) \quad \begin{cases} 
  y = -2x + 11 \\
  7x - 9y = 1 
\end{cases} & \quad (2) \quad \begin{cases} 
  4x + 3y = 7 \\
  3y = -7x + 10 
\end{cases}
\end{align*}
\]

Both the method of addition and subtraction, and the substitution method can be used to solve a set of simultaneous equations, but in both cases we solve by first eliminating one unknown.

Problem 7  Solve the following simultaneous equations, using any suitable method.

\[
\begin{align*}
(1) \quad \begin{cases} 
  -3x + 4y = 6 \\
  9x - 8y = -18 
\end{cases} & \quad (2) \quad \begin{cases} 
  y = 3x - 1 \\
  x - 2y = 12 
\end{cases}
\end{align*}
\]

\[
\begin{align*}
(3) \quad \begin{cases} 
  y = x + 1 \\
  y = -2x + 13 
\end{cases} & \quad (4) \quad \begin{cases} 
  3x - 2y = 12 \\
  2y = x - 8 
\end{cases}
\end{align*}
\]

Window on math — What balances with a bunch of bananas?

A bunch of bananas and three persimmons balance with 10 mandarin oranges.

The bunch of bananas also balances with six mandarin oranges and one persimmon.

In this case, how many mandarin oranges does the bunch of bananas balance with?
More Simultaneous Equations

Let's try solving simultaneous equations including parentheses, and simultaneous equations with coefficients that are not integers.

Example 1
Solve the following simultaneous equations.
\[
\begin{align*}
4x + y &= 10 \\
5x - 2(3x - y) &= -7
\end{align*}
\]

Hint: Multiply out and simplify as follows before solving.
\[
\begin{align*}
4x + y &= 10 \\
-x + 2y &= -7
\end{align*}
\]

Problem 1
Solve the simultaneous equations of Example 1.

For simultaneous equations including fractions or decimals, we can convert all coefficients to integers before solving.

Example 2
Solve the following simultaneous equations.
\[
\begin{align*}
4x + 3y &= -1 \\
\frac{1}{2}x - \frac{1}{3}y &= 2
\end{align*}
\]

Hint: Multiply both sides of (2) to cancel the denominator, giving as follows.
\[
\left( \frac{1}{2}x - \frac{1}{3}y \right) \times 6 = 2 \times 6 \\
3x - 2y = 12
\]

Problem 2
Solve the simultaneous equations of Example 2.

Problem 3
Solve the following simultaneous equations.
\[
\begin{align*}
(1) \quad &\begin{cases} 
5x + 2y = 1 \\
3x - 4(x + y) = 7
\end{cases} \\
(2) \quad &\begin{cases} 
3x + 2y = 6 \\
\frac{1}{4}x + \frac{2}{3}y = -1
\end{cases}
\end{align*}
\]
Example 3 Solve the following simultaneous equations.

\[
\begin{align*}
0.6x + 1.1y &= 7 \\
2x - y &= 14
\end{align*}
\]

Hint Multiply both sides of (1) by 10 to make all the coefficients into integers.

Problem 4 Solve the simultaneous equations of Example 3.

Problem 5 Solve the following simultaneous equations.

\[
\begin{align*}
0.2x + 0.3y &= -0.2 \\
5x + 2y &= 17
\end{align*}
\]

Basic Exercises

1 Which of the following pairs of values for \(x\) and \(y\) is the solution to the simultaneous equations:

- \(x = 1, \ y = 7\)
- \(x = -2, \ y = 1\)
- \(x = -1, \ y = 4\)
- \(x = 2, \ y = 1\)

2 Solve the following simultaneous equations.

- \(x = 2, \ y = 1\)
- \(x = 3, \ y = 9\)
- \(x = 4, \ y = 3\)
- \(x = 5, \ y = 2\)
- \(x = 1, \ y = 2\)
- \(x = 3, \ y = 14\)

See p. 44 Simultaneous equations of the form \(A = B = C\) (development)
Albert is taking part in a school cleaning project. He collected aluminum and steel cans. All in all he collected 25 cans with a total weight of 760 g. An aluminum can weighs 16 g. A steel can weighs 52 g. How many each of the aluminum and steel can did he collect?

Let's solve the above problem, using simultaneous equations.

Let the number of aluminum cans be $x$, and the number of steel cans be $y$.

There are a total of 25 cans, so

$$x + y = 25 \quad \text{(1)}$$

And the total weight is 760 g, so

$$16x + 52y = 760 \quad \text{(2)}$$

**Problem 1** Solve (1) and (2) as simultaneous equations, to find the solution to the question.

**Check 1** I bought roses at 200 yen each and carnations at 150 yen each, for a total cost of 1350 yen. I bought a total of eight flowers.

How many roses and how many carnations did I buy?

Approach the above problem in the following way.

1. Assign variables for the unknown quantities.
2. Create equations to represent the relation between the numbers of flowers, and the prices.
3. Solve the equations you created in step 2 to answer the original problem.
Example 1

The entrance to an art gallery costs 2800 yen for three junior high school students and five adults. It costs 1700 yen for two junior high school students and three adults. What are the admission charges per junior high school student and per adult?

Hint

The relations between quantities in the problem are as follows.

(junior high school student admission) \times 3 + (adult admission) \times 5 = 2800

(junior high school student admission) \times 2 + (adult admission) \times 3 = 1700

Answer

Let junior high school student admission be \( x \) yen, and adult admission be \( y \) yen, then:

\[
\begin{align*}
3x + 5y &= 2800 \\
2x + 3y &= 1700
\end{align*}
\]

\( (i) \times 2 \) \quad 6x + 10y = 5600

\( (2) \times 3 \) \quad 6x + 9y = 5100

\( (2) \times 3 \) \quad 6x + 9y = 5100

\( - \) \quad 6x + 9y = 5100

\( (i) \times 2 \) \quad 6x + 10y = 5600

Substitute (3) into (1) to give

\[
3x + 6 \times 500 = 2800
\]

\[
3x = 300
\]

\[
x = 100
\]

Answer: Junior high school student admission is 100 yen, and adult admission is 500 yen.

Problem 2

The total cost of five buns and four doughnuts is 890 yen, while the total cost of six buns and three doughnuts is 870 yen. Find how much a bun costs and how much a doughnut costs.

2 — Using Simultaneous Equations 37
Example 2: Ichiro walked 14 km from town A to town B for 4 hours. He walked uphill from Town A at 3 km/h. He then walked downhill to Town B at 5 km/h.

a) Find the distance from Town A to the top of the hill.

b) Find the distance from the top of the hill to Town B.

Hint: Let the distance from A to the top be $x$ km, and the distance from the top to B be $y$ km. Then we can represent the situation in example 2 using the figure below.

From the figure above, create simultaneous equations, and answer the question of Example 2.

In Example 2, let the time taken from A to the pass be $x$ hours, and the time taken from the pass to B be $y$ hours, then solve the problem.

Problem 5: Arthur leaves the house at 10 o'clock, and sets off for the station, 1200 m away. At first he walks at 50 m/minute, but fearing that he will miss the train, at some point he starts running at 200 m/minute, arriving at the station at 10:18.

Find how far he walked, and how far he ran.
Let's look at some proportion problems.

Example 3

In a certain Junior High School, there are 130 second year students. Of these, 15% of the boys and 10% of the girls take part in a volunteer activity: a total of 16 students.

Find how many boys and girls there are in the second year.

Hint

If the number of second year boys is \( x \), and the number of girls is \( y \), then the relations among the quantities are as shown in the following table.

<table>
<thead>
<tr>
<th>Total number ( )</th>
<th>Boys</th>
<th>Girls</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x )</td>
<td>( y )</td>
<td></td>
<td>130</td>
</tr>
<tr>
<td>Number taking part in volunteer activity ( )</td>
<td>( x \times \frac{15}{100} )</td>
<td>( y \times \frac{10}{100} )</td>
<td>16</td>
</tr>
</tbody>
</table>

From the above table, we obtain the following equations.

For the total number of students \( x + y = 130 \) \( \quad \) .......... (1)

For the number taking part \( \frac{15}{100} x + \frac{10}{100} y = 16 \) \( \quad \) .......... (2)

Problem 6

Solve (1) and (2) as simultaneous equations, to find the solution to Example 3.

Problem 7

I bought a volleyball and a soccer ball. The total retail price was 5500 yen, but since the volleyball was on sale at 90% of the standard price, and the soccer ball at 80%, the total cost came to 4600 yen. Find the original prices for a volleyball and a soccer ball.

\( \rightarrow p.43 \) Density of brine

2 — Using Simultaneous Equations
Basic Exercises

Using simultaneous equations

There are cans of soft drink that cost 120 yen each and 130 yen each. Buying a total of eleven of these cans, the cost is 1360 yen. Find how many of each type of can were in the purchase.

Use the following approach.

1. Assign variables to the unknown quantities.
2. Create equations to represent the relation between the numbers of cans, and the cost.
3. Solve the equations you created in step 2 to answer the original problem.

Window on math — Cranes and turtles

The following problem is from the third century Chinese classic by Sun Zi, where it appeared in the following form.

A basket contains pheasants and rabbits. There are a total of 35 heads and 94 legs. How many each of pheasants and rabbits are there?

This problem was later brought to Japan, where the pheasants and rabbits became cranes and turtles, and then problems like this became known in Japan as "Cranes and turtles problems".

1. Solve the above problem.
1. Solve the following simultaneous equations.

   \[ \begin{align*}
   3x + 2y &= 5 \\
   x - 2y &= 7 \\
   4x - 7y &= -6 \\
   6x + 2y &= -9 \\
   y &= 4x - 2 \\
   y &= x + 4
   \end{align*} \]

   \[ \begin{align*}
   6x - y &= 1 \\
   3x - 2y &= -7 \\
   y &= 5 + x \\
   5x - 2y &= 2 \\
   7x - 5y &= 17 \\
   8x + 3y &= 63
   \end{align*} \]

2. If the solutions of the simultaneous equations \[ \begin{align*}
   ax - by &= -13 \\
   bx + ay &= 1
   \end{align*} \] are \( x = -1 \) and \( y = 2 \), find the values of \( a \) and \( b \).

3. I bought a combination of 50-yen and 80-yen postage stamps, a total of 15 stamps. I gave a 1000-yen bill, and received 40 yen change. How many 50-yen and 80-yen stamps did I buy?

4. I have a two-digit natural number. Subtracting twice the units digit from three times the tens digit gives a difference of 1. The number formed by interchanging the digits is 9 larger than the original number. Find the original number.

5. In a class, \( \frac{1}{5} \) of the boys and \( \frac{1}{8} \) of the girls wear glasses. \( \frac{1}{6} \) of the 36 students wear glasses.

   Find how many boys and how many girls wear glasses.
1. Solve the following simultaneous equations.
   \[ \begin{align*}
   2x - 5y &= 20 \\
   -3(x - y) + y &= -2 \\
   \end{align*} \]

2. In the athletics club in a school, there were a total of 35 members last year. This year the girls have increased by 20%, while the boys have decreased by 20%, leading to a reduction in the members by just one. Find how many girls and how many boys there are in the club this year.

3. A lake has a circumference of 8 km. Ami and Brian start from the same point, Ami on a bicycle, and Brian walking, in opposite directions around the lake. If Ami and Brian set out at the same time, they meet after 30 minutes. If Ami sets out 20 minutes after Brian, she meets Brian 25 minutes after setting out. Find the speeds of Ami and Brian, in kilometers per hour.

Let's investigate!

Try to create a problem to be solved using simultaneous equations, in which one of the equations is \( x + y = 12 \).
An 8% salt solution, means for every 100 g of the solution there are 8 g of salt in the mixture.

How many grams of salt are present in 300 g of an 8% salt solution?

Let's look at a problem involving mixing salt solutions of different concentrations.

We mix $x$ g of 8% brine solution with $y$ g of 6% brine solution to make 500 g of 6% brine solution. Find how many grams of each of the solutions were mixed.

Use the following approach.

1. Write expressions in the blanks in the following table.

<table>
<thead>
<tr>
<th>Concentration</th>
<th>8%</th>
<th>3%</th>
<th>6%</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight of solution (g)</td>
<td>$x$</td>
<td>$y$</td>
<td>500</td>
</tr>
<tr>
<td>Weight of salt (g)</td>
<td>$x \times \frac{8}{100}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2. The total weight of solution and the total weight of salt remain the same before and after mixing. Use this fact to create equations.

3. Solve the equations you created in step 2 to answer the problem.
Simultaneous Equations of the Form \( A = B = C \)

The total entrance fee to a certain museum is the same at 250 yen, for two adults and one child or for one adult and three children. Find the admission fee to the museum for an adult and for a child.

In the trigger question, if we let the cost for an adult be \( x \) yen, and the cost for a child be \( y \) yen, then we have:

- Total admission cost for two adults and one child \((2x + y)\) yen
- Total admission cost for one adult and three children \((x + 3y)\) yen

Since both of these are 250 yen, we can write equations as follows:

\[
2x + y = x + 3y = 250 \quad \text{......................................................... (1)}
\]

When we have simultaneous equations in the form \( A = B = C \) as in (1),

\[
\begin{align*}
A &= B \\
A &= C \\
B &= C
\end{align*}
\]

we can solve any of the combinations:

**Problem 1**
For the simultaneous equations given by (1), try solving various different combinations, and check that you get the same answer in all cases.

**Problem 2**
Solve the following simultaneous equations.

1. \(4x + y = 3x - y = 7\)
2. \(x + y + 8 = 5x + y = 3x - y\)

44 2 — Simultaneous Equations
Sassa-tate

The following problem is a game using go stones found in an old Japanese text. Try it with your friends. Then think about how to obtain the solution, as below.

Take 30 go stones. Saying "Sah" each time, take either two or three stones from the 30. Place them on the left if you took two, on the right if you took three. After you have finished dividing the go stones, you can tell how many are on the left and how many on the right from the number of times you said "Sah".

If you said "Sah" eleven times, how many stones are there, left and right?

Deducing the answer

Number on left: \( 2 \times (\text{count of "Sah" calls} \times 3 - \text{initial number of stones}) \)

Number on right: \( 3 \times (\text{initial number of stones} - \text{count of "Sah" calls} \times 2) \)
In the following examples (1) to (4), we will investigate how one quantity varies when another quantity varies.

Draw a table of corresponding values of $x$ and $y$ in the different situations below. Try to see how the values of $y$ change as the values of $x$ change.

1. The time is $x$ minutes from starting to fill a tank with water, and the depth of water is $y$ cm

   
   ![Images of a tank being filled with water at different times.

   $$
   \begin{array}{c|c|c|c|c}
   x & 0:00 & 5:00 & 10:00 & 15:00 \\
   \hline
   y & & & & \\
   \end{array}
   $$

2. The width $x$ (cm) and height $y$ (cm) of a rectangle of area 18 cm$^2$

   
   ![Image of a rectangle with measurements.

   $$
   \begin{array}{c|c|c}
   x & & \\
   \hline
   y & & \\
   \end{array}
   $$

   

46 3 — Linear Functions
3) The weight $x$ (g) and the length of the spring $y$ (cm), when a weight is hung from a spring.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4) The time elapsed $x$ (minutes) from lighting an incense stick, and the remaining length of the incense stick, $y$ cm

How long will it take for the incense stick to burn completely?

<table>
<thead>
<tr>
<th>0:00</th>
<th>5:00</th>
<th>10:00</th>
<th>15:00</th>
</tr>
</thead>
<tbody>
<tr>
<td>cm</td>
<td>cm</td>
<td>cm</td>
<td>cm</td>
</tr>
</tbody>
</table>
1 Functions

In example (1) on page 46, we take the time the faucet was on to be \(x\) minutes, and the depth of water to be \(y\) cm.
Then for a certain value of \(x\) there is a corresponding value of \(y\).
Similarly in examples (2) to (4), fixing the value of \(x\) determines a value for \(y\).

Given two quantities represented by variables \(x\) and \(y\). If fixing \(x\) determines a single corresponding value of \(y\), we say that \(y\) is a function of \(x\).

In example (1), the depth of the water is a function of the time from starting to fill the tank.

Problem 1 Following the expression above, describe the situations in examples (2) to (4), using "... is a function of ...".

Example 1 In the following examples (1) and (2), show that \(y\) is a function of \(x\).

1. If I walk at 60 meters per minute for \(x\) minutes, I will have traveled \(y\) meters.
2. It takes \(y\) hours to walk a distance of 6 km at a speed of \(x\) km/h.

For cases (1) and (2) in Example 1 we can write the following expressions:

1. \(y = 60x\)
2. \(y = \frac{6}{x}\)

Case (1) is a direct proportionality, and case (2) is an inverse proportionality. The direct proportionality and inverse proportionality that you learned about in year 1 are also functions.

48 3 — Linear Functions
Example 2 I send a standard parcel within the same prefecture. If the sum of the length, width, and height of the parcel is \( x \) cm, and this determines the cost as \( y \) yen, then fixing the value of \( x \) determines a single value of \( y \). Therefore \( y \) is a function of \( x \).

<table>
<thead>
<tr>
<th>Sum of dimensions</th>
<th>Cost</th>
<th>Sum of dimensions</th>
<th>Cost</th>
</tr>
</thead>
<tbody>
<tr>
<td>up to 60 cm</td>
<td>600 yen</td>
<td>up to 140 cm</td>
<td>1,400 yen</td>
</tr>
<tr>
<td>up to 80 cm</td>
<td>800 yen</td>
<td>up to 160 cm</td>
<td>1,600 yen</td>
</tr>
<tr>
<td>up to 100 cm</td>
<td>1,000 yen</td>
<td>up to 170 cm</td>
<td>1,700 yen</td>
</tr>
<tr>
<td>up to 120 cm</td>
<td>1,200 yen</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

"Sum of dimensions of a standard parcel and cost"

Problem 2 The sum of the dimensions of two parcels are given below. What is the sending cost for each parcel?

1. 70 cm
2. 140 cm

In example 1 on page 46, if we let the time the faucet was turned on to be \( x \) minutes, and the depth of water to be \( y \) cm, then the relation \( y = 2x \) holds. This means that if we know the time we can find the depth of water in the tank. Thus, if we know the function that relates two variables \( x \) and \( y \), we can find the value of \( y \) from the value of \( x \).

Graphs in step form

For some functions, such as the one in Example 2 above, the value of \( y \) changes in steps. When we plot the graph of such a function, we get a step graph as shown on the right.

Problem 1 Find other examples in everyday life of functions with step graphs.
Linear Functions

Let's look at properties of the functions (1) to (4) given on pages 46 and 47.

In example (3) on page 47, consider the following.

1. For each increase by 1 g in the weight, by how many centimeters does the spring extend?
2. If we attach a weight of x g, how many centimeters is the extension of the spring?

In example (3), the length of the spring with no weight attached is 9 cm, and for each 1 g increase in the weight, the spring stretches by 0.2 cm. If the length of the spring is y cm when a weight of x g is attached, we can write y in terms of x as follows:

\[ y = 0.2x + 9 \]

For two variables x and y, if y can be written as a linear expression in x, we say that y is a linear function of x.

In general, we can write a linear function as follows:

\[ y = ax + b \]

where y is the dependent variable, x is the independent variable, a is the coefficient of the independent variable, and b is the constant term.

Example 1:
A cylindrical water tank contains water to a depth of 5 cm. Water is added to this tank so as to raise the water level by 2 cm every minute. If the depth of water in x minutes is y cm, then we can write y in terms of x as follows:

\[ y = 2x + 5 \]

Since this is of the form \( y = ax + b \), y is a linear function of x.
In example 4 on page 47, the length of the incense stick is originally 14 cm. Then when it is lit it becomes 0.4 cm shorter every minute. If the length of the incense stick in \( x \) minutes after being lit is \( y \) cm, then we can write \( y \) in terms of \( x \) as follows.

\[ y = 14 - 0.4x \]

In other words,

\[ y = -0.4x + 14 \]

So again this is of the form \( y = ax + b \), and \( y \) is a linear function of \( x \).

After \( x \) minutes

Problem 2 In Example 2, find how long it takes for the incense stick to be completely consumed after lighting.

Check 1 A certain car consumes 1 liter of gasoline for every 10 km. This car is filled with 50 l of gasoline. Answer the following if after \( x \) km the remaining gasoline is \( y \) in liters.

1. How many liters of gasoline are needed to travel 1 km?
2. Write an expression for \( y \) in term of \( x \).
3. After traveling 180 km, how many liters of gasoline are left?

Let's ask whether \( y \) is a linear function of \( x \) in examples (1) and (2) on page 46.

In (1), \( y = 2x \), and \( y \) is proportional to \( x \). This is a special case of a linear function written in the form \( y = ax + b \), in which the constant \( b \) is zero.

The equation \( y = ax \), indicating proportionality, is a special case of a linear function.

In (2), we have \( y = \frac{18}{x} \), and \( y \) is inversely proportional to \( x \). In this case, \( y \) is not represented by a linear expression, and therefore is not a linear function.
Changes in the Value of a Linear Function

Let's investigate changes in the value of a linear function.

Example 1

For the linear function \( y = 2x + 3 \), let's investigate the changes in the values of \( x \) and \( y \) as \( x \) is increased from 1 to 6.

The increase in the value of \( x \) is:

\[
6 - 1 = 5
\]

The increase in the value of \( y \) is:

\[
(2 \times 6 + 3) - (2 \times 1 + 3) = 10
\]

Here the increase in \( y \) is twice the increase in \( x \).

\[
\frac{\text{increase in } y}{\text{increase in } x} = 2
\]

Check 1

For the linear function \( y = 2x + 3 \), as \( x \) is increased from 3 to 7, find:

\[
\frac{\text{increase in } y}{\text{increase in } x}
\]

Problem 1

For the linear function \( y = -3x - 2 \), find:

\[
\frac{\text{increase in } y}{\text{increase in } x}
\]

as \( x \) is increased in the following cases.

(1) From 1 to 4  
(2) From 6 to 2

We call the ratio of the increase in the value of \( y \) to the increase in the value of \( x \) the rate of change.

\[
\text{rate of change} = \frac{\text{increase in } y}{\text{increase in } x}
\]

Problem 2

For the linear function \( y = 2x + 3 \), choose for yourself the values from and to which \( x \) is increased, and find the rate of change in this case.

Compare the rate of change you find with the results of Example 1 and Check 1.
Rate of change of a linear function

For the linear function \( y = ax + b \), the rate of change is equal to \( a \).

\[
\text{(rate of change)} = \frac{\text{(increase in } y \text{)}}{\text{(increase in } x \text{)}} = a
\]

From the above equation, we derive the following equation:

\[
\text{(increase in } y \text{)} = a \times \text{(increase in } x \text{)}
\]

Thus the increase in \( y \) is proportional to the increase in \( x \).

Moreover, the constant value \( a \) is the increase in the \( y \) for an increase of 1 in \( x \).

Check 2

For the following linear functions, find the rate of change.

If the increase in \( x \) is \( 1 \), find the increase in \( y \).

1. \( y = 3x + 5 \)
2. \( y = -\frac{1}{2}x - 1 \)

Problem 3

When the temperature on the ground is 15°C, up to about 10 km altitude, the temperature \( T \) °C at an altitude of \( x \) km is given by:

\[
y = -6x + 15
\]

1. What does the rate of change \(-6\) mean?
2. How many degrees does the temperature fall going from an altitude of 1 km to an altitude of 4 km?

Problem 4

In the inverse proportion \( y = \frac{24}{x} \), find the rate of change when \( x \) is increased in the following cases.

1. From 2 to 6
2. From 4 to 8

As you found in Problem 4, for inverse proportionality the rate of change is not constant.
4 Graphs of Linear Functions

Based on the linear function $y = 2x + 3$, let's try drawing a graph.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
<tr>
<td>$-4$</td>
<td>$-5$</td>
</tr>
<tr>
<td>$-3$</td>
<td>$-3$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-1$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$3$</td>
</tr>
<tr>
<td>$1$</td>
<td>$5$</td>
</tr>
<tr>
<td>$2$</td>
<td>$7$</td>
</tr>
<tr>
<td>$3$</td>
<td>$9$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$x$</th>
<th>$y$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\cdots$</td>
<td>$\cdots$</td>
</tr>
</tbody>
</table>

We'll draw points on the graph on the left with the coordinates corresponding to the pairs of $x$ and $y$ values in the table above.

Then we'll add some more points to the graph on the left.

Problem 1 For $y = 2x + 3$, take each value of $x$ from $-4$ to $3$ at intervals of 0.5, and find the corresponding value of $y$. Then add a point to the graph on the left at the coordinates for the pair of $x$ and $y$ values.

By adding more points, the graph becomes a straight line as shown in the diagram on the left.

This graph is the collection of all points having $(x, y)$ coordinates such that $y = 2x + 3$ holds.

Problem 2 Each of the following points lies on the graph of the linear function $y = 2x + 3$. Fill in the blue box with the missing value.

$A(6, \hspace{1cm})$  $B(\hspace{1cm}, 5)$  $C(\hspace{1cm}, 17)$
Next we'll compare the graphs of the two following linear functions.

\[ y = 2x \]  \hspace{1cm} (1)
\[ y = 2x + 3 \]  \hspace{1cm} (2)

We can combine the \( x \) and \( y \) values for both (1) and (2) in a single table, as follows.

| \( x \) | -4 | -3 | -2 | -1 | 0 | 1 | 2 | 3 | ...
|-------|----|----|----|----|---|---|---|---|------
| \( 2x \) | -8 | -6 | -4 | -2 | 0 | 2 | 4 | 6 | ...
| \( 2x + 3 \) | -5 | -3 | -1 | 1 | 3 | 5 | 7 | 9 | ...

As you will see from the above table, whatever the value of \( x \), the value for \( y \) in (2) is just 3 more than the value for \( y \) in (1).

Therefore, the points on the graph of (2) are the points on the graph of (1) shifted up by exactly 3.

Problem 3 ) Draw the following pairs of graphs of linear functions. In each case, for the (b) graph, say how much it is shifted up or down from the (a) function.

(1) \[ \begin{align*}
(\text{a}) & \quad y &= -2x \\
(\text{b}) & \quad y &= 2x + 5 \\
\end{align*} \]

(2) \[ \begin{align*}
(\text{a}) & \quad y &= \frac{1}{2}x \\
(\text{b}) & \quad y &= \frac{1}{2}x - 4 \\
\end{align*} \]
The graph of the linear function \( y = ax + b \) is the straight line made by shifting the graph of \( y = ax \) up parallel by the amount \( b \) in the positive \( y \)-direction.

Note: If \( b \) is negative, for example if \( b = -5 \), then shifting the graph parallel by 5 in the positive \( y \)-direction means the same as shifting the graph parallel by -5 in the negative \( y \)-direction.

Check 1: The straight line in the diagram on the left is the graph of \( y = -x \). Based on this graph, draw the graph of \( y = -x - 2 \).

The constant part \( b \) of the linear function \( y = ax + b \) is the value of \( y \) when \( x = 0 \), and the \( y \)-coordinate of the point \((0, b)\) where the graph intersects the \( y \)-axis. This \( b \) is called the **y-intercept** of the linear function \( y = ax + b \).

Check 2: Give the y-intercept of the linear function \( y = 2x + 4 \).

The graph of \( y = ax + b \) is a straight line passing through the point \((0, b)\), and parallel to the graph of \( y = ax \).
The Slope

We'll investigate what is meant by "rate of change" for a graph.

For the linear function \( y = 2x + 3 \), the rate of change is 2, and therefore:

\[
\frac{\text{increase in } y}{\text{increase in } x} = 2
\]

In terms of the graph, moving to the right by 1 means moving up by 2.

**Problem 4** For the graph of \( y = 2x + 3 \), what is the movement up for a movement to the right of 3?

In the same way, the rate of change of the linear function \( y = -2x + 5 \) is \(-2\), and therefore, for a movement to the right of 1 on the graph there is a movement down of 2. In other words, for a movement to the right of 1 on the graph there is a movement up of \(-2\).

**Problem 5** For the graph of \( y = -2x + 5 \), for a movement to the right of 4, what is the movement down?

As you will see from what we have learned already, the inclination at which the graph of the linear function \( y = ax + b \) lies is determined by \( a \). We call \( a \) the slope of the graph.

- **Graph of a linear function**

  For the graph of the linear function \( y = ax + b \), the slope is \( a \) and the y-intercept is \( b \).
The slope $a$ is the amount by which $y$ increases for an increase of $x$ by 1.

When $a > 0$

When $a < 0$

Example 1 For the graph of the linear function $y = 4x - 5$,
the slope is 4 and the $y$-intercept is $-5$.

Check 3 For the following linear functions, give the slope and $y$-intercept of the graph.

1) $y = 3x + 4$
2) $y = -2x - 1$

Problem 6 For the following linear functions, give the slope and $y$-intercept of the graph.

1) $y = x - 2$
2) $y = -4x$
3) $y = \frac{3}{2}x - 6$

Taking $y = 2x + 3$ as an example, we can show the relations among the table, equation, and graph as follows.
Let's try drawing the graph of a linear function, based on the y-intercept and slope.

**Example 2**

Let's draw the graph of the linear function \( y = -\frac{1}{2}x + 3 \).

Since the y-intercept is 3, the graph intersects the y-axis at (0, 3). And since the slope is \(-\frac{1}{2}\), if we move right by 4 units, from the point (0, 3), for example, we must move down by 2 units to the point (4, 1), which lies on the graph.

Therefore all we need to do is to draw a straight line through the points (0, 3) and (4, 1).

Choosing points further apart makes it easier to draw the graph accurately.

---

**Check 4**

Draw the graph of the linear function

\[ y = -2x - 1 \]

**Problem 7**

Draw the graphs of the following linear functions.

1. \( y = x + 1 \)
2. \( y = -2x - 2 \)
3. \( y = \frac{1}{3}x + 2 \)

**Problem 8**

Of the graphs you drew in Problem 7, say which are straight lines sloping up to the right.
For the linear function \( y = ax + b \), we can say the following:

1. **When \( a > 0 \)**
   - When \( x \) increases, \( y \) also increases.
   - The graph is a straight line sloping up to the right.

2. **When \( a < 0 \)**
   - When \( x \) increases, \( y \) decreases.
   - The graph is a straight line sloping down to the right.

The graph of \( y = 2x + 3 \), and we can call this:
- the straight line \( y = 2x + 3 \)
We also call \( y = 2x + 3 \) the equation of this straight line.

We will find the equation of a straight line by finding the slope and y-intercept of the graph.

**Example 1** Find the equation of straight line (1) in the diagram on the right.

- The y-intercept is 3.
- Moving 3 to the right along the line moves up 1, and therefore the slope is \( \frac{1}{3} \).
- Therefore the equation of the line is
  \[
  y = \frac{1}{3}x + 3
  \]

**Check 5** Find the equation of straight line (2) in the diagram above.

**Problem 9** Find the equations of straight lines (3) and (4) in the diagram above.
Linear Function Graphs and Variable Ranges

In the linear function \( y = 2x - 1 \), find the values of \( y \) when \( x = 3 \) and \( x = 5 \).

From the graph of a linear function, we'll find the range of the variables.

For the linear function \( y = 2x - 1 \), when \( 3 \leq x \leq 5 \), the range of the variable \( x \) is shown by a red line segment on the \( x \)-axis, and the range of the variable \( y \) is shown by a red line segment on the \( y \)-axis.

In other words, for the linear function \( y = 2x - 1 \), if the range of \( x \) is \( 3 \leq x \leq 5 \), the range of \( y \) is \( 5 \leq y \leq 9 \).

Problem 10 For the linear function \( y = 2x - 1 \), answer the following questions.

1. Draw the graph of this function.
2. Find the values of \( y \) corresponding to \( x = -1 \) and \( x = 3 \).
3. If the range of \( x \) is \(-1 < x < 3\), find the range of \( y \).
Finding Linear Functions

When Given the Rate of Change and a Pair of \( x \) and \( y \) Values

Example 1
Find the linear function for which the rate of change is \(-2\), and when \( x = 3, y = 2 \).

\[
\text{Since the rate of change is } -2, \text{ this linear function is of the form } \ y = -2x + b
\]

\[
\text{When } x = 3, y = 2, \text{ and therefore: } 2 = -2 \times 3 + b
\]

\[
\text{Solving this gives } b = 8 \quad \text{Answer: } y = -2x + 8
\]

Example 1 is the same as finding the linear function whose graph has a slope of \(-2\) and passes through the point \((3, 2)\).

Check 1
Find the linear functions that satisfy the following sets of conditions.

1. The rate of change is \(3\), and when \(x = 1, y = 4\).
2. The slope of the graph is \(-3\), and the graph passes through the point \((1, 2)\).

Problem 1
Find the linear functions that satisfy the following sets of conditions.

1. The rate of change is \(-1\), and when \(x = -2, y = -3\).
2. The graph passes through the point \((2, 0)\), and is parallel to the straight line \(y = 2x + 5\).

You can use the same method when given the coordinates of one point, and the y-intercept of the graph.

Problem 2
Find the linear function whose graph passes through the point \((2, 1)\) and has a y-intercept of 5.

For \(y = ax + b\), this is when you know the value of \(b\).
When Given Two Pairs of $x$ and $y$ Values

**Example 1**

Find the linear function such that when $x = 2$, $y = 3$, and when $x = 5$, $y = 9$.

**Answer**

The slope of a straight line through the two points $(2, 3)$ and $(5, 9)$ is:

$$\frac{9 - 3}{5 - 2} = 2$$

Therefore

$$y = 2x + b$$

Substitute $x = 2$ and $y = 3$ into this equation to find the value of $b$:

$$3 = 2 \times 2 + b$$

$$b = -1$$

Answer $y = 2x - 1$

We can also solve Example 2 by creating a pair of simultaneous equations and solving.

**Answer**

Let the linear function we are looking for be $y = ax + b$

When $x = 2$, $y = 3$, and therefore:

$$3 = 2a + b$$

(1)

When $x = 5$, $y = 9$, and therefore:

$$9 = 5a + b$$

(2)

Solving equations (1) and (2) for $a$ and $b$, we get

$a = 2$, $b = -1$

Answer $y = 2x - 1$

**Check 2**

Find the linear function such that when $x = 2$, $y = 3$, and when $x = 4$, $y = -9$.

**Problem 3**

Find the linear function whose graph passes through the points $(−3, 5)$ and $(3, −1)$.
Basic Exercises

1. **Linear functions**
   - Water is added at a constant rate to a tank containing 2 liters of water. Three minutes after starting to add the water, the amount in the tank is 11 liters.
   - How many liters are added each minute?
   - Write an expression for $y$ in terms of $x$, if $y$ liters is the amount of water in the tank $x$ minutes after starting to add the water.

2. **Rate of change of linear functions**
   - Give the rate of change of the linear function $y = 4x + 1$.

3. **Slope and $y$-intercept of a graph**
   - Give the slope and $y$-intercept of the linear function $y = 5x - 3$.

4. **Graphs of linear functions**
   - Draw the graphs of the following linear functions:
     - $y = 3x - 4$
     - $y = \frac{1}{3}x + 2$

5. **Equation for a straight line**
   - Find the equations for straight lines (1) and (2) in the diagram on the right.

6. **Finding linear equations**
   - Find the linear functions that satisfy the following sets of conditions.
     - The rate of change is 3, and when $x = 1$, $y = -1$.
     - When $x = -3$, $y = 3$, and when $x = 3$, $y = 5$.
The following equation is a linear equation in two unknowns, \( x \) and \( y \).

\[ x + 2y - 2 = 0 \]  \hspace{1cm} (1)

For equation (1), we’ll find the values of \( y \) corresponding to values of \( x \), and complete the blanks in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -5 )</th>
<th>( -4 )</th>
<th>( -3 )</th>
<th>( -2 )</th>
<th>( -1 )</th>
<th>( 0 )</th>
<th>( 1 )</th>
<th>( 2 )</th>
<th>( 3 )</th>
<th>( 4 )</th>
<th>( 5 )</th>
<th>...</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>

If we plot the points whose coordinates are the values of \( x \) and \( y \) in the above table, we get the result in the diagram on Graph 1 below. As we add more and more of the points whose coordinates are values of \( x \) and \( y \), we get the graph shown in the diagram on Graph 2 below.

In equation (1), fixing the value of \( x \) determines the value of \( y \), and therefore \( y \) is a function of \( x \). To show this function more clearly, we solve equation (1) for \( y \), to get:

\[ y = -\frac{1}{2}x + 1 \] \hspace{1cm} (2)
The graph shown on the previous page is the graph of the linear function (2), with a slope of $-\frac{1}{2}$ and a y-intercept of 1.

We call this straight line the graph of the equation $x + 2y - 2 = 0$.

The graph of the equation $x + 2y - 2 = 0$ is the set of points whose $x$ and $y$ coordinates are pairs of values satisfying the equation.

---

Let's draw the graphs of linear equations in two unknowns.

**Example 1**

Let's draw the graph of the equation $3x - 4y - 12 = 0$.

Solving this equation for $y$ gives:

$$y = \frac{3}{4}x - 3$$

Therefore the graph is a straight line with slope $\frac{3}{4}$ and y-intercept $-3$.

---

Check 1

Draw the graph of the equation $2x + y - 3 = 0$.

Problem 1

Draw the graphs of the following equations.

1. $x + 2y = -4$
2. $3x - 2y + 8 = 0$
We can also use the following method to draw the graph of a linear equation in two unknowns.

**Example 2**

Draw the graph of the equation \(2x - 3y + 6 = 0\).

**Hint**

Given two points, we can draw a straight line through them.

When \(x = 0\), \(y = 2\)

When \(y = 0\), \(x = -3\)

Therefore the graph passes through the two points 
\((0, 2), (-3, 0)\)

**Answer**

![Graph of \(2x - 3y + 6 = 0\)]

The calculation is easier setting \(x\) and \(y\) to zero.

**Check 2**

Draw the graph of the equation \(2x - y - 6 = 0\).

**Problem 2**

Draw the graphs of the following equations.

1. \(x + 3y = -6\)
2. \(2x - 5y + 10 = 0\)
3. \(\frac{x}{4} + \frac{y}{3} = 1\)
For the linear equation in two unknowns \( ax + by + c = 0 \), let's draw the graph when the coefficient \( a \) is zero.

**Example 3**

Draw the graph of \( 2y - 6 = 0 \).

This is a linear equation in two unknowns of the form \( 0x + 2y - 6 = 0 \).

Thus whatever the value of \( x \), it is always true that
\[
2y - 6 = 0, \text{ or } y = 3
\]

This graph is the set of points whose \( y \)-coordinate is 3, and thus the points such as
\[
(-1, 3), (0, 3), (1, 3), (2, 3)
\]

all lie on this graph. Therefore the graph is a straight line parallel to the \( x \)-axis and passing through the point \((0, 3)\).

---

Check 3

Draw the graph of the equation \( 5y = 10 \).

Problem 3

Draw the graphs of the following equations.

1. \(-3y + 3 = 0\)
2. \(2y = -8\)
Simultaneous Equations and Graphs

Let's solve the following simultaneous equations.

\[
\begin{align*}
2x - y &= 1 \\
x + y &= 5
\end{align*}
\]

The set of points whose coordinates are \((x, y)\) such that \(x\) and \(y\) satisfy equation (1) above, is straight line (1) shown in the diagram on the right. The set of points whose coordinates are \((x, y)\) such that \(x\) and \(y\) satisfy equation (2) above, is straight line (2).

Therefore, the common point with coordinates that satisfy both of the simultaneous equations above is the intersection of the graphs (1) and (2).

Problem 1) Read off the coordinate of the intersection of the graphs in the diagram above, and check that the \(x\)- and \(y\)-coordinates are the solution to the above simultaneous equations.

---

Solutions of simultaneous equations and Intersections of graphs

The solution of simultaneous equations in \(x\) and \(y\) is the pair of \(x\) and \(y\) coordinates of the intersection of the graphs of the equations.

Check 1) Find the solutions of the following simultaneous equations by drawing the graphs.

\[
\begin{align*}
3x + y &= 2 \\
2x - y &= 3
\end{align*}
\]
We can find the coordinates of the intersection of two straight lines by solving the pair of equations for the straight lines as simultaneous equations.

**Problem 2** Find the coordinates of the intersection of the two straight lines in the diagram on the right, using the following procedure.

1. Find the equations for lines (1) and (2).
2. Solve the simultaneous equations you found in step 1, to find the coordinates of the intersection.

**Problem 3** The graph of \( y = 2x - 3 \) intersects the \( x \)-axis at point A. Find the coordinates of point A.

Finding the coordinates of the point where a graph intersects the \( x \)-axis is the same as finding the coordinates of the intersection of the graph and the straight line \( y = 0 \).

Let's try!

Try solving the following simultaneous equations, using graphs.

\[
\begin{align*}
\text{1} & \quad \begin{cases} 2x - y = 1 \\ 4x - 2y = 8 \end{cases} \\
\text{2} & \quad \begin{cases} 2x - y = 1 \\ 4x - 2y = 2 \end{cases}
\end{align*}
\]
Using Linear Functions

Water was heated in an experiment. The temperature of water was measured every after 1 minute for 5 minutes. The table below shows the result of the experiment. \( x \) is the time in minutes and \( y \) is the water temperature in °C.

We will predict how long will it take for the water temperature to be 70 °C.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>23.0</td>
<td>28.4</td>
<td>34.2</td>
<td>39.3</td>
<td>45.0</td>
<td>50.6</td>
</tr>
</tbody>
</table>

Since the points plotted in the diagram on the right all lie more or less on a straight line, we can see that \( y \) is a linear function of \( x \).

**Problem 1**

Consider the points plotted in the diagram on the right to lie on a straight line passing through the points \((0, 23)\) and \((4, 45)\). Draw this line, and answer the following questions.

1. Find the equation for the straight line.
2. What do the slope and y-intercept of the straight line represent?
3. Can you estimate how long it will take for the water temperature to reach 70 °C?

As we considered above, by treating the results of an experiment as a simple function, we can understand the basic relation between two quantities.
Geometrical Shapes and Linear Functions

In the diagram of rectangle ABCD on the right, point P starts out from A, and moves along the sides through points B and C, to D.

When point P has moved \( x \) cm from A, let the area of \( \triangle APD \) be \( y \) cm\(^2\); we will investigate the way in which the area of \( \triangle APD \) changes.

Trigger When P is moving along side AB, let's find the equation for \( y \) in terms of \( x \).

Next we'll think about when point P is moving along side BC.

As in the diagram on the right, when point P moves along side BC, the area of \( \triangle APD \) is constant, equal to 6 cm\(^2\).

Therefore, When \( 3 \leq x \leq 7 \), \( y = 6 \)

Problem 2 When P is moving along side CD, write an expression for \( y \) in terms of \( x \).

How do you represent the length of DP using \( x \)?

Problem 3 Draw a graph showing how the area of \( \triangle APD \) changes as point P moves along sides AB, BC, and CD.
Using Graphs of Linear Functions

The diagram below is a graph showing the movement of trains between two stations P and Q which are 8 km apart, from 9 o'clock to 10 o'clock.

---

Let's think about more aspects of the graph above.

**Example 1**
Andrew sets out from station P at 9:05 on a bicycle, traveling at 12 km/hour alongside the track to station Q. How many trains coming from station Q will he meet before reaching station Q himself?

**Hint**
Draw a graph on the diagram above showing Andrew's progress from station P to station Q.

**Problem 4**
Using the above hint, find the solution to Example 1.

**Problem 5**
In Example 1, how many trains coming from station P will pass Andrew?

---

Try to make a problem to be solved using the above graph.
Barbara leaves the house at 9 o'clock, and sets off by bicycle for the park 12 km away. For the first 30 minutes she rides at 12 km/hour, she then takes a rest, and finally sets off at 18 km/hour, reaching the park at exactly 10 o'clock.

3) Five minutes after starting her rest, Barbara is passed by her sister, also on her bicycle on the way from the house to the park, and Barbara and her sister arrive at the park at the same time. If Barbara's sister traveled at a constant speed, how many minutes after nine o'clock did she leave the house?

Let's try /

A line segment AB has length 40 cm. Points P and Q move, starting from A at the same time, and each traveling to and fro along AB at a constant speed. One two-way journey takes P 20 seconds, and takes Q 6 seconds. During a single two-way journey of P from A and back again, how many times will P and Q cross at intermediate points.
1. Draw the graphs of the following equations.
   ① \(2x + y = 4\)
   ② \(3y + 9 = 0\)

2. Find the solutions of the following simultaneous equations by drawing graphs.
   \[
   \begin{align*}
   2x + y &= 4 \\
   2x - 3y &= 12
   \end{align*}
   \]

3. Clive leaves the house at 10 AM, and goes by bicycle to town A. From town A he walks to town B. The graph on the right shows the relation between time elapsed from leaving the house and his distance traveled. In this example, answer the following questions.
   ① Find his speed, when he traveled by bicycle from his house to town A.
   ② At 10:20, his sister Dorothy sets out from the house on her bicycle at 18 km/hour, to catch up with her brother. Find the time when she reaches him by drawing a graph. How many kilometers from the house is the point where they meet?
1. For the linear equation \( y = -3x + 5 \), answer the following questions.
   
   (1) Find the values of \( y \) when \( x = -3 \) and \( x = 2 \).
   
   (2) Find the increase in the value of \( y \) when the value of \( x \) is increased by 5.
   
   (3) If the range of variable \( x \) is \(-4 \leq x \leq 3\), find the range of variable \( y \).

2. Find the linear equations meeting the following sets of conditions.
   
   (1) When \( x = 5, y = 3 \), and when \( x \) is increased by 5, \( y \) increases by 2.
   
   (2) The graph passes through points \((2, 3)\) and \((-5, -11)\).
   
   (3) The graph passes through point \((-1, 5)\), and is parallel to the straight line \( y = -3x - 5 \).

3. Answer the following questions about the diagram on the right.
   
   (1) Find the equations for lines \( \text{C} \) to \( \text{E} \).
   
   (2) Find the solution of the following simultaneous equations, using the diagram on the right.
      
      \[
      \begin{cases}
      x - 2y = -2 \\
      x + y = 7
      \end{cases}
      
   (3) Find the coordinates of the intersection of lines \( \text{F} \) and \( \text{G} \).

4. In a certain town, the charge for water supply is a linear function of the amount of water used, in the range from 21 m³ to 30 m³, inclusive. A family uses 22 m³ in May and is charged 2880 yen, then in June uses 26 m³ and is charged 3580 yen.

   In July, if they use 28 m³, what will the water charge be?
The tank shown in the diagram below is a rectangular parallelepiped, with a depth of 1 m. Water is supplied to this tank at a constant rate from a pipe A, and drains at a constant rate from pipe B. The pipes are opened and closed according to the following sequence 1 to 3.

1. When the tank is full, pipe A closes, and water just drains from pipe B.
2. When the water draining from pipe B brings the water depth to 20 cm, pipe B closes, and simultaneously pipe A opens, supplying water.
3. When the water supplied from pipe A brings the water depth to 80 cm, pipe B opens, and water is both supplied and drained simultaneously.

The graph on the right shows part of the change in the depth of water in the tank. How long does it take for one cycle from the tank being full until the next time it is full?

**Let’s investigate!**

Kazuko goes from town A to town B. The graph on the right shows the approximate relation between time and the distance she has traveled. Choose suitable numbers, and tell a story about how she went.
Graph of $x = h$

On page 68, we looked at the graph of the linear equation in two unknowns,

$$a x + b y + c = 0$$

when $a$, the coefficient of $x$, is zero.

Now we ask: What does the graph look like when $b$, the coefficient of $y$, is zero?

**Example** 1 Let's draw the graph of $3x - 6 = 0$.

$3x - 6 = 0$ is an equation in two unknowns of the form

$3x + 0y - 6 = 0$. Therefore, for any value of $y$, $3x - 6 = 0$, or $x = 2$, holds.

This graph is the set of points for which the $x$-coordinate is 2, or in other words,

the straight line parallel to the $y$-axis passing through the point $(2, 0)$.

### Problem 1

Draw the graphs of the following equations in the diagram on the left.

1. $x = -2$
2. $-x + 3 = 0$
3. $2y = 10$

If $a$, $b$, and $c$ are constants, the graph of the linear equation in two unknowns, $ax + by + c = 0$, is as follows:

When $a = 0$, a straight line parallel to the $x$-axis

When $b = 0$, a straight line parallel to the $y$-axis
Dividing the oil

I have 10 liters of oil. Using only a 3-liter container and a 7-liter container, how can I divide this equally into two parts of 5 liters?

The figures in the diagram below each consist of three congruent shapes, squares and equilateral triangles, joined together. Divide each of these figures into four congruent parts.
For each of these polygons, we’ll find the sum of all the angles.

- Quadrilateral
- Pentagon
- Hexagon
- Heptagon
- Octagon

I wonder if we can find the answer without measuring every angle?
You learned in elementary school that the sum of the three angles of a triangle is 180°. Using this fact, let's think of a way to find the sum of the angles of a polygon of four or more sides.

Let's try dividing the polygon into triangles.
Choose your own way of drawing lines to divide the pentagon on the right into triangles.
Using the same method, try dividing all of the polygons on the opposite page.

Fill in the blanks in the following table, to find the sum of the angles of each polygon.

<table>
<thead>
<tr>
<th>Quadrilateral</th>
<th>Pentagon</th>
<th>Hexagon</th>
<th>Heptagon</th>
<th>Octagon</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of triangles</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Sum of angles</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Interior Angles and Exterior Angles of a Polygon

1. Sum of the Interior Angles

The diagonals from any vertex of a polygon divide the polygon into a number of triangles. Let's use this fact to find the sum of the angles.

Problem 1: Use the above division to find the sum of the angles of the polygon. Fill in the blanks in the following table, and make an expression for the sum of the angles of a polygon.

<table>
<thead>
<tr>
<th>Polygon</th>
<th>Number of vertices</th>
<th>Number of diagonals</th>
<th>Number of triangles</th>
<th>Sum of angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>Quadrilateral</td>
<td>4</td>
<td>1</td>
<td>2</td>
<td>$180^\circ \times 2$</td>
</tr>
<tr>
<td>Pentagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Hexagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heptagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Octagon</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

To refer to a polygon, we write the letters for the vertices in order around the polygon. For example, we write "pentagon ABCDE." As in $\angle BAP$ in the diagram on the right, we call the angle between the extension of one side and the adjacent side as the exterior angle of that vertex. Angles such as $\angle BAE$ and $\angle ABC$ are called interior angles. In the diagram on the right, if we extend side BA, and create another exterior angle of the vertex A, this angle is equal to $\angle BAP$.

Here when we say "polygon" we are considering only a convex polygon, and excluding polygons as shown on the right, which include a concavity.
If \( n \) is the number of vertices of a polygon, we can find the sum of the interior angles with the following expression.

\[
\text{The sum of the interior angles of a polygon with } n \text{ sides is } 180^\circ \times (n - 2)
\]

**Example 1** Find the sum of the interior angles of a decagon (10-sided polygon).

In the above expression, substitute 10 for \( n \) to obtain:

\[
180^\circ \times (10 - 2) = 1440^\circ
\]

Therefore the sum of the interior angles of a decagon is 1440°.

**Check 1** Find the sum of the interior angles of a dodecagon (12-sided polygon).

**Problem 2** Answer the following questions about the interior angles of a polygon.

1. Find the size of each interior angle of a regular nonagon (9-sided polygon).
2. How many vertices has a polygon whose interior angles sum to 1620°?

Let’s try!

Take a point that is not a vertex of a polygon, and join this point to each vertex, to make a number of triangles. Using these triangles, try to find the sum of the interior angles of the polygon.
Sum of the Exterior Angles

A jogging track is as shown in the diagram on the right. Alfie starts from point P, proceeding in the direction of the arrow. After completing one circuit, he returns to P. Through how many degrees has he rotated the direction he is facing?

Let's consider the sum of the exterior angles* of a polygon, starting from the expression for the sum of the interior angles.

First consider a pentagon.
At each vertex the sum of the interior and exterior angles is $180^\circ$. Therefore the total of all the interior and exterior angles for the five vertices is:

$$180^\circ \times 5 = 900^\circ$$

But the sum of the interior angles is

$$180^\circ \times (5 - 2) = 540^\circ$$

Therefore the sum of the exterior angles of a pentagon is

$$900^\circ - 540^\circ = 360^\circ$$

Problem 3 In the same way as above, find the sum of the exterior angles of a quadrilateral and hexagon.

---

*By the sum of the exterior angles is meant the sum of one angle for each vertex.
The sum of the exterior angles of a polygon is 360°

Example 2) In the diagram on the right, find the angle shaded in color.

Hint: Since the sum of the exterior angles of a polygon is 360°, the shaded angle is

$$360° - (80° + 70° + 115°) = 95°$$

Answer: 95°

Check 2) In the following diagram, find the size of shaded angles.

Problem 4) Answer the following questions about the exterior angles of a regular polygon.

1) Find the size of one exterior angle of a regular octagon.

2) How many sides has a regular polygon of which the exterior angles are each 30°?

Let's try!

Explain the fact that the sum of the exterior angles of a polygon with $n$ sides is 360°, using the variable $n$. 
Parallel Lines and Angles

In Section 1 we started from the fact that the sum of the interior angles of a triangle is 180°, and derived various properties of the angles of a polygon. You checked this property of the sum of interior angles of a triangle using a protractor in elementary school, but let's see if we can derive this from more basic properties, as we did in Section 1.

To do this, first we will investigate basic properties of geometrical figures.

Andrew and Belinda are playing on a seesaw. If Andrew goes down by 10°, by how many degrees does Belinda go up?

When two straight lines intersect, this forms angles around the point of intersection. Angles like \( \angle a \) and \( \angle c \) in the diagram on the right are called vertical angles. \( \angle b \) and \( \angle d \) are also vertical angles.

In the diagram on the right, whatever the size of \( \angle b \), the following are true.

\[
\angle a = 180° - \angle b \\
\angle c = 180° - \angle b
\]

Since \( \angle a \) and \( \angle c \) are each equal to \( 180° - \angle b \), then \( \angle a = \angle c \).

When indicating angles, we sometimes write just the name of the vertex, as in \( \angle E \), or indicate the angle with a lowercase letter, as \( \angle x \).
Problem 1) In the diagram on the previous page, explain why \( \angle b = \angle d \).

---

Properties of vertical angles

Vertical angles are equal.

---

Example 1) Three lines intersect at a point, as in the diagram on the right. In this case, since \( \angle a \) is vertically opposite the angle of \( 45^\circ \), we have \( \angle a = 45^\circ \).

---

Check 1 In the diagram of Example 1, find the sizes of \( \angle b, \angle c, \) and \( \angle d \).

In the diagram on the right, straight line \( n \) intersects two straight lines, line \( \ell \) and \( m \). We say that angles \( \angle a \) and \( \angle e \) are corresponding angles \( \angle b, \angle f, \angle c, \angle g, \) and \( \angle d, \angle h \) are also pairs of corresponding angles.

We say that angles such as \( \angle b \) and \( \angle h \) are alternate angles. \( \angle c \) and \( \angle e \) are also alternate angles.

Check 2 Answer the following questions about the diagram on the right.

1. Which angle is a corresponding angle with \( \angle p \)?
2. Which angle is an alternate angle with \( \angle s \)?

Try looking for the other corresponding angles and alternate angles.
Parallel Lines and Corresponding Angles

As shown in the diagram on the right, we'll draw two parallel lines using set squares, then mark corresponding angles.

If we draw two straight lines \( \ell \) and \( m \) so that corresponding angles with respect to another straight line \( n \) are equal, then \( \ell \) and \( m \) are parallel.

If we draw a straight line \( n \) to intersect two parallel straight lines \( \ell \) and \( m \), then the corresponding angles are equal.

In other words, in the diagram on the right the following are true.

1. If \( \angle a = \angle b \), then \( \ell \parallel m \).
2. If \( \ell \parallel m \), then \( \angle a = \angle b \).

Check 3: In the diagram on the right, when \( \ell \parallel m \), find the size of \( \angle a \).

Parallel Lines and Alternate Angles

In the diagram for Check 3 above, how big is \( \angle b \)?

In the diagram on the right, when the two straight lines \( \ell \) and \( m \) are parallel, then the alternate interior angles \( \angle a \) and \( \angle b \) are equal.

We can show why this is the case, as follows.
Since $\angle a$ and $\angle c$ are corresponding angles for a pair of parallel lines $\angle a = \angle c$.

Since $\angle b$ and $\angle c$ are vertical angles $\angle b = \angle c$.

Since both $\angle a$ and $\angle b$ are equal to $\angle c$ $\angle a = \angle b$.

Problem 2) In the diagram on the right, assume $\angle a = \angle b$. In this case explain why $\ell \parallel m$, using the fact that corresponding angles are equal.

We can summarize the relation between parallel lines and angles as follows.

<table>
<thead>
<tr>
<th>Relation between parallel lines and angles</th>
</tr>
</thead>
<tbody>
<tr>
<td>When one straight line intersects a pair of other straight lines</td>
</tr>
<tr>
<td>1 If the pair of straight lines are parallel, then corresponding angles and alternate angles are equal.</td>
</tr>
<tr>
<td>2 If corresponding angles or alternate angles are equal, then the pair of straight lines are parallel.</td>
</tr>
</tbody>
</table>

Check 4 Of the straight lines in the diagram on the right, indicate which are parallel, using the $\parallel$ symbol. Also indicate which sets of angles from $\angle x$, $\angle y$, $\angle z$, and $\angle u$ are equal.

Problem 3) In the diagram on the right, given that $\ell \parallel m$, find the size of $\angle x$ and $\angle y$. Also find the value of $\angle x + \angle y$.
**Proof**

In elementary school, you investigated the interior angles of a triangle by measuring them, and by rearranging the pieces of a triangle as shown on the right, and you learned that the sum of the interior angles of a triangle is $180^\circ$.

We can now explain why this is true, using the properties of parallel lines.

As shown in the diagram on the right, let $CD$ be the extension of side $BC$ of $\triangle ABC$. Then draw a straight line $CE$ passing through $C$ and parallel to side $AB$. In this case, the following hold:

Since alternate angles of parallel lines are equal

\[ \angle a = \angle a' \]

Since corresponding angles of parallel lines are equal

\[ \angle b = \angle b' \]

Therefore the sum of the interior angles of $\triangle ABC$ is

\[ \angle a + \angle b + \angle c = \angle a' + \angle b' + \angle c \]

\[ = 180^\circ \]

In the above explanation, based on the properties of parallel lines we deduced that the sum of the interior angles of a triangle is $180^\circ$. Showing that something must hold from an assumption of properties that we already know to be true is called proof.

From this proof, we can also derive the following fact:

\[ \angle ACD = \angle a + \angle b \]
Properties of the interior and exterior angles of a triangle

1. The sum of the interior angles of a triangle is 180°.
2. An exterior angle of a triangle is equal to the sum of the two other interior angles.

Example 2 In the following diagrams, find the size of \( \angle x \).

\[ \begin{align*}
(1) \quad \angle x &= 180° - (56° + 42°) = 82° \\
(2) \quad \angle x &= 47° + 38° = 85°
\end{align*} \]

Answer 82°
Answer 85°

Check 5 In the following diagrams, find the size of \( \angle x \).

\[ \begin{align*}
(1) \quad \angle x &= 180° - (63° + \text{other angle}) \\
(2) \quad \angle x &= 180° - (52° + \text{other angle}) \\
(3) \quad \angle x &= 180° - (128° + \text{other angle})
\end{align*} \]

Problem 4 In the diagram on the right, draw a straight line \( DE \) through vertex \( A \) of \( \triangle ABC \) parallel to \( BC \). Using this diagram, prove that the sum of the interior angles of a triangle is 180°.
Problem 5) In the following diagrams, find the size of $\angle x$. In (1), $\ell \parallel m$.

Let's try!

In the following diagrams, draw lines to collect the interior angles of the triangle about point P, and use this to prove that the sum of the interior angles of a triangle is 180°.

Window on math — Euclid and geometry

Around the third century BC, the Greek Euclid produced a collection of the known properties of geometrical figures, in his "Elements", comprising thirteen books. In this he started from a small number of very simple assumptions, and proceeded in sequence to derive many many more complicated properties. The Elements became a model textbook, and was read for more than 2000 years by people around the world.
Basic Exercises

1. Sum of the interior angles and exterior angles of a polygon

Answer the following questions about a regular pentagon.

1. Find the sum of the interior angles.
2. Find the size of an exterior angle.

2. Vertically opposite angles, corresponding angles, and alternate angles

In the diagram on the right, name the
given that is vertically opposite \( \angle b \), the
corresponding angle to \( \angle b \), and the alternate
angle to \( \angle b \).

3. Relation between parallel lines and angles

In the following diagram, if \( l \parallel m \), find the size of \( \angle x \).

1. \( l \parallel m \) with \( \angle 60 \)
2. \( l \parallel m \) with \( \angle 40 \)

4. Properties of interior and exterior angles of a triangle

In the following diagram, find the size of \( \angle x \).

1. \( \angle 73 \) and \( \angle 58 \)
2. \( \angle 131 \) and \( \angle 44 \)
The pattern on the right is based on a geometrical figure, and consists of a tiling of many figures, all congruent to this figure. What is this basic figure?

For two figures in the plane, if by sliding and flipping one figure we can align it to coincide with the other figure, then we say these figures are congruent.

Check 1 In the following diagram, which of the triangles (a) to (d) are congruent to \( \triangle ABC \)? Say which are the corresponding vertices and sides.

This is a traditional Japanese pattern, called "Hemp leaves". The photo on the right shows a hemp plant.
For congruent figures, the following facts hold.

**For congruent figures, the corresponding line segments and angles are equal.**

In the diagram on the right, suppose the quadrilaterals $ABCD$ and $A'B'C'D'$ are congruent, and the corresponding vertices are $A$ and $A'$, $B$ and $B'$, $C$ and $C'$, and $D$ and $D'$. In this case we write

$$\text{Quadrilateral } ABCD \equiv A'B'C'D'$$

The $\equiv$ symbol indicates congruence. When using this symbol, we write the symbols for the corresponding vertices in sequence around the quadrilaterals.

**Check 2** In Check 1 on the previous page, show which triangles are congruent with $\triangle ABC$, using the $\equiv$ symbol.

**Problem 1** If Pentagon $ABCDE \equiv$ Pentagon $FGHIJ$, say which are the corresponding sides and angles.

In congruent figures, since corresponding line segments and angles are equal, for example when $\triangle ABC \equiv \triangle A'B'C'$ we can say the following:

- $AB = A'B'$, $BC = B'C'$, $CA = C'A'$
- $\angle A = \angle A'$, $\angle B = \angle B'$, $\angle C = \angle C'$
Conditions for Congruent Triangles

Trigger

Hanako had the following conversation with her younger brother Jiro, who is at elementary school.

My homework is to draw a triangle with two sides of 5 cm and 6 cm, and one angle of 50°.

In the above diagram, if the two sides and the angle are in the relative positions shown in the diagram on the right, then even with the two sides and one angle fixed, there are two possibilities for the triangle.

On the other hand, if the angle is between the two sides (called the "included angle"), then fixing the two sides and the angle determines a unique triangle.

Problem 1

In the following cases, is there a unique triangle satisfying the conditions? Try actually drawing the triangles.

1. Triangle with sides 4 cm, 5 cm, and 6 cm
2. Triangle with one side 6 cm, and two angles of 50° and 60°

In case 1 in Problem 1, there is a unique triangle. Again, in case 2, if the length of one side, and the angles at both ends of this side are fixed, then there is only one possible triangle.
Therefore, for the following combinations of sides and angles, if we know their sizes, this determines a unique possible triangle.

(a) Three sides
(b) Two sides and the included angle
(c) One side and the angles at both ends

Given \( \triangle ABC \), using any of (a) to (c) above we can draw another triangle congruent to \( \triangle ABC \).

Since there is a unique triangle, the shape and size are both the same.

From what we have found out so far, we can derive the following congruence conditions for triangles.

<table>
<thead>
<tr>
<th>Congruence conditions for triangles</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Two triangles are congruent if any of the following sets of conditions hold.</strong></td>
</tr>
<tr>
<td><strong>1</strong> Three sides are the same.</td>
</tr>
<tr>
<td>( \triangle ABC ) &amp; ( \triangle A'B'C' )</td>
</tr>
<tr>
<td>( AB = A'B' )</td>
</tr>
<tr>
<td>( BC = B'C' )</td>
</tr>
<tr>
<td>( CA = C'A' )</td>
</tr>
<tr>
<td><strong>2</strong> Two sides and the included angle are the same.</td>
</tr>
<tr>
<td>( \triangle ABC ) &amp; ( \triangle A'B'C' )</td>
</tr>
<tr>
<td>( AB = A'B' )</td>
</tr>
<tr>
<td>( BC = B'C' )</td>
</tr>
<tr>
<td>( \angle B = \angle B' )</td>
</tr>
<tr>
<td><strong>3</strong> One side and the angles at each end of the side are the same.</td>
</tr>
<tr>
<td>( \triangle ABC ) &amp; ( \triangle A'B'C' )</td>
</tr>
<tr>
<td>( BC = B'C' )</td>
</tr>
<tr>
<td>( \angle B = \angle B' )</td>
</tr>
<tr>
<td>( \angle C = \angle C' )</td>
</tr>
</tbody>
</table>
To check whether two triangles are congruent, we do not always need to move the triangles around to see if they coincide, we can determine that they are congruent if any of the above sets of conditions hold.

**Example 1**

In the following diagram, for \( \triangle ABC \) and \( \triangle QPR \), we have

\[
AB = QP, \quad BC = PR, \quad CA = RQ
\]

And therefore, since all three sides are the same length

\[\triangle ABC = \triangle QPR\]

![Diagram of triangles](image)

**Check 1**

In the above diagram, find other congruent triangles in additions to the ones in Example 1, and show them using the \( = \) symbol. In each case, say which conditions you used.

**Problem 2**

In each of the following diagrams, find a pair of congruent triangles, and show them using the \( = \) symbol. In each case, say which conditions you used. In each diagram, sides and angles marked with the same symbol are equal.

![Diagram of triangles](image)
Proofs Using Triangle Congruence Conditions

In the first year, you learned the construction for bisecting an angle; now we will use the congruence conditions for triangles to prove that this construction is correct.

We can construct the bisector OC of \( \angle XOY \) as follows.

1. Draw a circle with the vertex O as center, and let A and B be the points of intersection of this circle with the lines OX and OY respectively.
2. Draw circles of the same radius with centers A and B, and let C be their point of intersection.
3. Draw the ray OC.

**Example 2** Prove that the ray OC constructed above is the bisector of \( \angle XOY \).

**Hint** As in the diagram on the right, join A to C, and B to C, and prove the result by showing that triangles \( \triangle OAC \) and \( \triangle OBC \) are congruent.

**Proof** In triangles \( \triangle OAC \) and \( \triangle OBC \),

\[
\begin{align*}
OA &= OB \\
AC &= BC \\
OC &= OC
\end{align*}
\]

Since the three sides are the same, we have

\( \triangle OAC \equiv \triangle OBC \)

Since corresponding angles of congruent triangles are equal,

\( \angle AOC = \angle BOC \)

Therefore, OC is the bisector of \( \angle XOY \).
Check 2 In the diagram on the right, if
\[ \begin{align*}
AB &= AD \\
\angle ABC &= \angle ADE
\end{align*} \]
then
\[ BC = DE \]

1. To prove this, which two triangles would we need to prove congruent?
2. Say which triangle congruence conditions are used in the proof in (1).

---

**Window on math — Constructing equal angles and parallel lines**

Let's investigate why constructions work.

1. As in the following diagram, we can construct \( \angle DAB \) to be equal to \( \angle XOY \). Look at the procedure for this construction.

   Then in this construction, prove that \( \angle XOY = \angle DAB \).

   ![Diagram](image)

2. The diagram on the right shows a method of construction of a line through point \( P \) not on the straight line \( l \), parallel to line \( \ell \). In the construction on the right, take a point \( A \) on \( \ell \), and find points \( B \) and \( Q \) such that \( AP = AB = PQ \), \( BP = AQ \).

   Prove that in this construction \( PQ \parallel \ell \).
Creating Proofs

Hypothesis and Conclusion

In the diagram on the right, a straight line \( n \) intersects two straight lines \( \ell \) and \( m \). Enter the appropriate symbol in the blank in the following statements.

1. If \( \ell \parallel m \), then \( \angle a \) \( \approx \angle b \)
2. If \( \angle c = \angle d \), then \( \ell \parallel m \)
3. If \( \ell \parallel m \), then \( \angle a + \angle d \) = 180°

Often we make statements about a figure in the following way:

\[ \text{something} \text{ then } \text{something} \]

In a statement like this, we call the part \[ \text{something} \] before "then" the hypothesis, and the part \[ \text{something} \] after "then" the conclusion.

Example 1: In (1) above, \( \ell \parallel m \) is the hypothesis, and \( \angle a = \angle b \) is the conclusion.

Check 1: For (2) and (3) above, say what is the hypothesis, and what is the conclusion.

Problem 1: For the following statements (1) and (2), say what is the hypothesis, and what is the conclusion.

1. If \( \triangle ABC \equiv \triangle DEF \), then \( AB = DE \).
2. If \( x \) is a multiple of 6, then \( x \) is a multiple of 2.
Reasoning Underlying a Proof

We'll make geometrical proofs, while writing out the reasons underlying the proof.

Example 2
In the diagram on the right, E is the point of intersection of line segments AB and CD, and:

\[ EA = EB, \ AD \parallel CB \]

Prove that \( ED = EC \).

\[ \text{Hint} \]

The hypothesis is: \( EA = EB, \ AD \parallel CB \)
The conclusion is: \( ED = EC \)

We can show that the conclusion follows from the hypothesis by showing that triangles \( \triangle AED \) and \( \triangle BEC \) are congruent.

\[ \text{Proof} \]

In \( \triangle AED \) and \( \triangle BEC \):

\[
\begin{align*}
EA &= EB & \text{Hypothesis} \\
\angle AED &= \angle BEC & \text{Vertical angles are equal} \\
\angle EAD &= \angle EBC & \text{Alternate angles of parallel lines are equal}
\end{align*}
\]

Therefore

\( \triangle AED \cong \triangle BEC \)

And thus

\( ED = EC \)

In this way, when proving something, we can use any facts we have already proved to be true.
Check 2 In the diagram on the right, if O is the midpoint of each of AB and CD, then AC // DB. We prove this using the following sequence of statements.

Say what the reason is for each of (1) to (4).

In \( \triangle AOC \) and \( \triangle BOD \):

\[
\begin{align*}
OA &= OB & \text{Hypothesis} \\
OC &= OD & \text{Hypothesis} \\
\angle AOC &= \angle BOD & \text{(1)}
\end{align*}
\]

Therefore

\( \triangle AOC \cong \triangle BOD \) \hspace{2cm} (2)

And thus

\( \angle OAC = \angle OBD \) \hspace{2cm} (3)

From which we get

\( AC \parallel DB \) \hspace{2cm} (4)

Problem 2 In the diagram on the right, when

\( AC = BD, \ AD = BC \)

then AC // DB. Answer the following questions about this fact.

(1) Give the hypothesis and conclusion.

(2) To derive the conclusion from the hypothesis, we can use the fact that two triangles in the diagram are congruent. Which triangles should we prove congruent?

(3) Give a proof, writing out the reasons behind each step.
Let's collect together facts from what we have learned so far that we will be able to use often as a basis for reasoning in a proof.

**Properties of vertically opposite angles**

Vertically opposite angles are equal.

**Relations between parallel lines and angles**

When a straight line intersects two straight lines:

1. If the straight lines are parallel, then corresponding angles and alternate angles are equal.
2. If corresponding angles and alternate angles are equal, then the straight lines are parallel.

**Properties of interior and exterior angles of a triangle**

1. The sum of the interior angles of a triangle is 180°.
2. An exterior angle of a triangle is equal to the sum of the two adjacent interior angles.
For a polygon with \( n \) sides, the sum of the interior angles is \( 180\degree \times (n - 2) \).

The sum of the exterior angles of a polygon is \( 360\degree \).

In congruent figures, corresponding line segments and angles are equal.

Two triangles are congruent if any of the following sets of conditions holds.

1. All three respective sides are equal.
2. Two sides and the included angle are equal.
3. One side and the angles at each end of the side are equal in the two triangles.

In addition to the above list, we can also use properties of equations, the formula for an area or volume, and so on as a basis for the reasoning of a proof.
Basic Exercises

1. Conditions for congruent triangles
   In the following cases, what conditions need to be added so that triangles ABC and DEF are congruent?
   1. AB = DE, \( \angle B = \angle E \)
   2. \( \angle A = \angle D, \angle C = \angle F \)
   3. AC = DF, BC = EF

2. Creating proofs - hypothesis and conclusion
   In the diagram on the right, if \( AB = DC \) and \( AC = DB \), then \( \angle BAC = \angle CDB \).
   We can use the sequence below to prove this. Answer the following questions.
   1. Give the hypothesis and conclusion.
   2. Fill in the blank below with the names of the appropriate triangles.
   3. Give the basis for the reasoning in steps (1) and (2) below.

   In \( \triangle ABC \) and \( \triangle DCB \):
   \[
   \begin{align*}
   AB &= DC \\
   AC &= DB \\
   BC &= CB
   \end{align*}
   \]
   Therefore \( \triangle ABC \equiv \triangle DCB \) ................... (1)
   And thus \( \angle BAC = \angle CDB \) ................... (2)
1. In the diagram on the right, if $\ell \parallel m$, and $\angle d = 40^\circ$, find the other sizes of the angles.

2. In the following diagrams, find the size of $\angle x$.

3. Answer the following questions.
   1. How many sides has a regular polygon with exterior angles of $45^\circ$?
   2. Find the sum of the interior angles of a polygon with 20 sides.
   3. How many sides has a regular polygon for which the sum of the interior angles is $1260^\circ$?

4. In the diagram on the right, if $PA = PB$ and $M$ is the midpoint of the line segment $AB$, then $\angle AMP = \angle BMP$. Answer the following questions.
   1. Give the hypothesis and conclusion.
   2. To derive the conclusion from the hypothesis, we can use the fact that two triangles in the diagram are congruent. Which triangles are these?
   3. In the proof of (2), give the conditions used for the congruence of the triangles.
Chapter Summary Problems \( \text{B} \)

1. In the diagram below, if \( \| m \), find the size of \( x \).

\[ \begin{align*}
\text{Diagram 1:} & \quad \angle 160^\circ, \angle 60^\circ \nonumber \\
\text{Diagram 2:} & \quad x = 30^\circ \\
\end{align*} \]

2. If \( P \) is a point on the perpendicular bisector of line segment \( AB \), then \( AP = BP \).

Give a proof of this, using conditions for congruence of two triangles.

3. In the diagram on the right, if \( AB = AD, \ BE = DC \)

Prove that \( BC = DE \)

---

Let's investigate!

What kind of book was Euclid's "Elements," referred to on page 92?

Choose some of the geometrical properties we have studied in this chapter, and try to find out how these are discussed in "Elements."

---

Euclid's "Elements"
In the diagram on the right, the following holds:

\[ \angle ADC = \angle A + \angle B + \angle C \quad \ldots \quad (1) \]

Let's use this relation to find the sum of angles.

1. **Give different explanations for why (1) holds.**

2. **Using relation (1), in each of the following diagrams, find the sum of the five marked angles.**

3. **In the diagram on the right, find the sum of the seven marked angles.**

4. **Try increasing the number of points on the circle, or changing the way they are connected, to draw different diagrams, then find the sum of the angles.**

Connecting seven points by joining every other point around the circle gives us the diagram on the right.
Chapter 5 Properties of Geometrical Figures

1 Triangles

The diagrams below show a method of making a right angle at the position of point A. Think about why this makes a right angle.
Remembering the method used on the previous page, try to make a right angle at the point A.

In the diagram you have drawn above, check that $\angle BAC$ is a right angle.

Is it always a right angle?
In the above diagram, let $\angle OAB = \angle a$, and $\angle OAC = \angle b$, then consider the sum of the interior angles of $\triangle ABC$. From this, prove that $\angle BAC = 90^\circ$.

In an isosceles triangle:
- We call the vertex between the two equal sides the apex.
- The side opposite the apex is the base.
- The angles at each end of the base are the base angles.

In the proof of Problem 1, we used the following facts.

1. A triangle with two sides equal is an isosceles triangle.
2. The base angles of an isosceles triangle are equal.
3. The sum of the interior angles of a triangle is $180^\circ$.

When we say precisely what a term means, as in (1) above. We call this a definition.
We proved statement (3) on the previous page using the relations between parallel lines and angles in Chapter 4. But you only checked statement (2) in elementary school by folding a paper triangle, or by measuring the angles.

Let's now prove statement (2) on the previous page:
Base angles of an isosceles triangle are equal.

Hint: If we consider \( \triangle ABC \) in which \( AB = AC \), then we have to derive that \( \angle B = \angle C \).

Give the hypothesis and conclusion for the statement "Base angles of an isosceles triangle are equal."

Proof: Construct the perpendicular bisector of the apex \( \angle A \), and let \( D \) be the point of intersection with the base.

Then in triangles \( \triangle ABD \) and \( \triangle ACD \):

\[
\begin{align*}
\angle B & = \angle C \\
AD & \text{ is common} \\
\angle BAD & = \angle CAD
\end{align*}
\]

Since these triangles have two sides and the included angle equal,

\( \triangle ABD \cong \triangle ACD \)

Therefore \( \angle B = \angle C \)

The above proof above applies to any \( \triangle ABC \) in which \( AB = AC \). In other words, for any isosceles triangle, the base angles are equal.
We often use properties such as "The base angles of an isosceles triangle are equal." or "The sum of the interior angles of a triangle is 180°." as a basis for proving other geometrical properties. Important facts like this that have been proved are called theorems.

**Theorem** Base angles of an isosceles triangle are equal

Using properties of the base angles of an isosceles triangle, let's investigate the size of the angles.

**Example 1** In the diagram on the right, if sides with the same marking are equal, find the size of \( \angle x \).

**Hint** \( \angle x = \frac{(180° - 100°)}{2} = 40° \)

**Answer** 40°

**Check 1** In each of the diagrams below, if sides with the same marking are equal, find the size of \( \angle x \).

An angle greater than 0° and less than 90° is called an acute angle; an angle greater than 90° and less than 180° is called an obtuse angle.

**Problem 2** Say why a base angle of an isosceles triangle is always acute.

An isosceles triangle whose apex is a right angle is called a right-angled isosceles triangle.
Properties of the Bisector of the Apex of an Isosceles Triangle

As shown in the diagram on the right, if we draw the bisector of the apex $\angle A$ of isosceles triangle $ABC$, and let $D$ be the point where this intersects $BC$, then:

\[ BD = CD, \quad AD \perp BC \]

Prove that this is the case.

As we proved on page 113,
\[ \triangle ABD \cong \triangle ACD \]

The fact that $BD = CD$ follows immediately, because they are corresponding sides of congruent triangles.

Problem 3: Fill in \[ \boxed{\quad} \] in the following, to complete the proof that $AD \perp BC$.

Since $\triangle ABD \cong \triangle ACD$, corresponding angles are equal, and thus

\[ \angle ADB = \boxed{\quad} \quad \text{\ldots (1)} \]
\[ \angle ADB + \boxed{\quad} = 180^\circ \quad \text{\ldots (2)} \]

From (1) and (2), $2 \angle ADB = 180^\circ$

Therefore,
\[ \angle ADB = \boxed{\quad} \]

And thus
\[ AD \perp BC \]

From our study above, we get the following theorem.

**Theorem** The bisector of the apex of an isosceles triangle perpendicularly bisects the base.
**Problem 4** In the diagram on the right, suppose that $CA = CB$ and $DA = DB$.

1. Prove that $\angle ACD = \angle BCD$.
2. From the result of (1), prove that $CD$ is the perpendicular bisector of line segment $AB$.

---

**Equilateral Triangles**

The definition of an equilateral triangle is

A triangle whose three sides are all equal

Let's prove that the interior angles of an equilateral triangle are all equal. For this, in $\triangle ABC$, from hypothesis $AB = BC = CA$

we have to derive

collection $\angle A = \angle B = \angle C$

---

**Problem 5** Fill in the blanks in the following with angles, to complete the proof.

Since we can think of $\triangle ABC$ as an isosceles triangle with $AB = AC$:

$\angle B = \underline{\angle \phantom{B}}$ ........................................ (1)

Similarly we can think of $\triangle ABC$ as an isosceles triangle with $BA = BC$, so:

$\angle A = \underline{\angle \phantom{A}}$ ........................................ (2)

From (1) and (2), $\angle A = \angle B = \angle C$
2 Conditions for an Isosceles Triangle

What conditions have to be applied to make a triangle isosceles?

If you fold a piece of paper tape as shown below, what sort of triangle is the overlapping triangular portion?

In the above, we can derive the fact that $\angle ABC = \angle ACB$ from looking at the tape as a pair of parallel lines.

Problem 1) In $\triangle ABC$ of the above, explain why $\angle ABC = \angle ACB$.

When two sides of a triangle are equal, two angles are also equal .................. (a)
We have already proved this. But in the reverse direction, can we also say
If two angles of a triangle are equal, two sides are ......................... (b) also equal

Let's prove that if a triangle has two angles equal, the two sides containing the angles are also equal, for $\triangle ABC$. From the hypothesis $\angle B = \angle C$ we have to derive the conclusion $AB = AC$.
Let's now prove that if a triangle has two angles equal two sides are also equal.

Proof > Draw the bisector of \( \angle A \), and let its intersection with \( BC \) be point \( D \).

In \( \triangle ABD \) and \( \triangle ACD \),

\[ \angle B = \angle C \]
\[ \angle BAD = \angle CAD \quad (1) \]

Since the sum of the interior angles of a triangle is 180°, the remaining angles are also equal.

Therefore \( \angle ADB = \angle ADC \quad (2) \)

And \( AD \) is common \( \quad (3) \)

From (1), (2), and (3), since one side and the angles at each end are all equal,

\[ \triangle ABD \cong \triangle ACD \]

And therefore, \( AB = AC \)

---

**Conditions for an isosceles triangle**

**Theorem** If two angles of a triangle are equal, the triangle is an isosceles triangle, with the equal angles as the base angles.

---

**Using the above theorem, let's investigate some geometrical properties.**

**Check 1** In isosceles triangle \( ABC \), draw the bisectors of \( \angle B \) and \( \angle C \), and let \( P \) be the point of their intersection. Now prove that \( \triangle PBC \) is an isosceles triangle.

**Problem 2** A triangle with all three angles equal is an equilateral triangle. Prove this.
Converse of a Theorem

Let's compare the property of the base angles of an isosceles triangle (a) and the condition for an isosceles triangle (b) on page 117.

For \( \triangle ABC \), we can write (a) and (b) respectively as:

(a) If \( AB = AC \) then \( \angle B = \angle C \)
(b) If \( \angle B = \angle C \) then \( AB = AC \)

Comparing (a) and (b) we find that the hypothesis and conclusion have changed places.

When we take a theorem and interchange the hypothesis and conclusion, we state the converse of the theorem.

In other words If \( \bigcirc \bigcirc \bigcirc \) then \( \bigcirc \bigcirc \bigcirc \)
the converse is If \( \bigcirc \bigcirc \bigcirc \) then \( \bigcirc \bigcirc \bigcirc \)

Above, (b) is the converse of (a), and (a) is also the converse of (b).

**Check 2** In the diagram on the right, give the converse of If \( \ell \parallel m \) then \( \angle a = \angle b \).

**Problem 3** Give the converse of each of the following statements. Also say whether each one is true.

1. For an equilateral triangle, the three interior angles are equal.
2. If \( x \geq 5 \), then \( x > 3 \)

As you found in Problem 3, the converse of a true statement is not necessarily true. Therefore, before saying that the converse of a theorem is true, it is first necessary to give a proof.
Let's investigate the conditions of the sides and angles of two right-angled triangles for them to be congruent.

In $\triangle ABC$ and $\triangle DEF$, if
\[
\begin{align*}
\angle C &= \angle F = 90^\circ \\
AB &= DE \\
\angle A &= \angle D
\end{align*}
\]
then can we say that $\triangle ABC \equiv \triangle DEF$?

In a right-angled triangle, the side opposite the right angle is called the hypotenuse.

As you will see from the question, two right-angled triangles are congruent if the hypotenuse and one acute angle in each triangle are equal.

If two right-angled triangles have the hypotenuse and one other side equal, then they are congruent. In other words:

In $\triangle ABC$ and $\triangle DEF$, if
\[
\begin{align*}
\angle C &= \angle F = 90^\circ \\
AB &= DE \\
AC &= DF
\end{align*}
\]
then $\triangle ABC \equiv \triangle DEF$

Let's prove this.
Since \( AC = DF \), we can flip over \( \triangle DEF \) and place \( AC \) and \( DF \) together. Since \( \angle C = \angle F = 90^\circ \), \( \angle BCE = 180^\circ \), and the three points \( B, C, \) and \( E \) lie on a straight line.

**Problem 1** In \( \triangle ABE \) above, explain why \( \angle B = \angle E \). Using this fact, give a proof that \( \triangle ABC \cong \triangle DEF \).

---

**Congruence conditions for right-angled triangles**

**Theorem** Two right-angled triangles are congruent if either of the following conditions holds:

1. The hypotenuse and one acute angle are equal.
2. The hypotenuse and one other side of each triangle are equal.

Compare these with the congruence conditions for triangles in general.

**Check 1** In the diagram below, which are the congruent triangles? Express this using the \( \cong \) symbol. Give the conditions that you used to justify saying that the triangles are congruent.
Using the congruence conditions for right-angled triangles, let's look at more geometrical properties.

Problem 2) In \( \triangle ABC \), drop perpendiculars from \( M \), the midpoint of side \( BC \) to sides \( AB \) and \( AC \) respectively. Let the points where these perpendiculars intersect \( AB \) and \( AC \) be \( D \) and \( E \) respectively. If \( MD = ME \), then \( \triangle ABC \) is an isosceles triangle. Prove this.

Let's look at the properties of the bisectors of the interior angles of a triangle.

Problem 3) As in the diagram on the left, let \( I \) be the point of intersection of the bisectors of \( \angle B \) and \( \angle C \) of \( \triangle ABC \), and drop perpendiculars from \( I \) to the three sides. Let the points of intersection of these perpendiculars with \( AB, BC, \) and \( CA \) be \( D, E, \) and \( F \) respectively.

1. Prove that \( ID = IE = IF \).
2. Prove that the ray \( AI \) bisects \( \angle BAC \).

Problem 4) In the diagram of Problem above, draw a circle with \( I \) as center and \( IE \) as radius.
Properties of isosceles triangles

1. In the following diagrams, sides marked with the same symbol are equal.
   Find the size of \( \angle x \).

   \[ \begin{array}{c}
   \text{(1)} \quad 82^\circ \\
   \text{(2)} \quad 42^\circ \\
   \text{(3)} \quad 117^\circ 
   \end{array} \]

Converse of a theorem

2. Give the converse of: "If \( \triangle ABC \cong \triangle DEF \), then \( AB = DE \)." Say whether this converse is true.

Congruence conditions for right-angled triangles and conditions for an isosceles triangle

3. In the diagram on the right,
   \[ BE = CD \]
   \[ \angle BEC = \angle CDB = 90^\circ \]
   In this case, answer the following questions.
   (1) Give the congruence conditions used to prove that \( \triangle BCF \cong \triangle CBD \).
   (2) To prove that \( AB = AC \), what do we need to show from \( \triangle BCF \cong \triangle CBD \).
Parallelograms

Properties of a Parallelogram

As shown on the right, lay a piece of tape over another piece. What shape is the area where one piece covers another?

The sides that face each other in a quadrilateral are called opposite sides, and the diagonally opposite angles in a quadrilateral are called opposite angles.

The definition of a parallelogram is:

A quadrilateral with both pairs of opposite sides equal

We sometimes write the parallelogram ABCD as \( \square ABCD \).

In \( \square ABCD \) on the right, if the diagonals intersect at point O, show which lines and angles are equal.

From the definition of a parallelogram and basic geometrical properties we can derive the following properties.

<table>
<thead>
<tr>
<th>Theorems</th>
<th>Properties of a parallelogram</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>The two pairs of opposite sides of a parallelogram are equal.</td>
</tr>
<tr>
<td>2</td>
<td>The two pairs of diagonally opposite angles of a parallelogram are equal.</td>
</tr>
<tr>
<td>3</td>
<td>The two diagonals of a parallelogram intersect at their midpoints.</td>
</tr>
</tbody>
</table>

**Hint**: The hypothesis that quadrilateral ABCD is a parallelogram is:

- $AB \parallel DC$, $AD \parallel BC$

From this we have to derive the conclusion:

- $AB = DC$, $AD = BC$

To do this, we draw the diagonal $AC$, and show that:

- $\triangle ABC \equiv \triangle CDA$

**Proof**: Draw the diagonal $AC$.

In $\triangle ABC$ and $\triangle CDA$

- $AC$ is common \(1\)

Since alternate angles of parallel lines are equal

- $\angle BCA = \angle DAC \quad \quad \quad \quad \quad \quad 2$
- $\angle BAC = \angle DCA \quad \quad \quad \quad \quad \quad 3$

Since one side and the adjacent angles are equal

- $\triangle ABC \equiv \triangle CDA$

Therefore

- $AB = DC$, $AD = BC$

For \( \square ABCD \), taking point \( O \) as the intersection of the diagonals, let's prove property 3 of a parallelogram.

**Problem 2** For the parallelogram in the diagram on the right, give the hypothesis and conclusion for property 3.

We can write a proof of property 3 as follows.

**Proof**

In \( \triangle ABO \) and \( \triangle CDO \)

Since opposite sides of a parallelogram are equal

\[
AB = CD \quad (1)
\]

And since alternate angles of parallel lines are equal

\[
\angle BAO = \angle DCO \quad (2)
\]

\[
\angle BAO = \angle DCO \quad (3)
\]

Since one side and the adjacent angles are equal

\( \triangle ABO \equiv \triangle CDO \)

Therefore

\[
OA = OC, \ OB = OD
\]

**Check 1** For \( \square ABCD \) in (1) and (2) below, find the values of \( x \) and \( y \).

Say what properties of a parallelogram you used to find the values.

\( x = 60^\circ \)

\( y = 120^\circ \)
Let's see what we can prove using properties of a parallelogram.

Example 1: Let O be the point of intersection of the diagonals of \(\square ABCD\). Let E and F be the points of intersection of a line passing through O with AD and BC respectively, as shown in the diagram on the right. Then \(OE = OF\). Prove this.

Can you see what we need to prove in order for \(OE = OF\)?

Proof:

In \(\triangle AOE\) and \(\triangle COF\)

Since the diagonals of a parallelogram intersect at their midpoints

\[OA = OC\] \hspace{1cm} (1)

And since diagonally opposite angles are equal

\[\angle AOE = \angle COF\] \hspace{1cm} (2)

Since alternate angles of parallel lines are equal

\[\angle EAO = \angle FCO\] \hspace{1cm} (3)

Since one side and the adjacent angles are equal

\[\triangle AOE = \triangle COF\]

Therefore

\[OE = OF\]

Check 2: On the diagonal BD of \(\square ABCD\), if we choose two points E and F such that \(BE = DF\). Prove that \(AE = CF\).
2 Conditions for a Parallelogram

...In the previous section we derived various properties of a parallelogram. Now we will consider what conditions are needed to say that a figure is a parallelogram.

In the kind of fairground ride in the photographs, the seating platform moves while remaining horizontal. Why is this?

The diagram on the right shows the ride above, seen from the side. In this diagram, when the seating platform AD moves, what parts do not change?
In the diagram at the bottom of the previous page, by joining points B and C we can consider the quadrilateral ABCD as in the diagram on the right. In the diagram on the right, AD is the seating platform, and AD moves while keeping AB = DC and AD = BC. Since BC is horizontal, to prove that the seating platform remains horizontal we need to show that AD // BC.

In other words, for the quadrilateral ABCD, from the hypothesis AB = DC, AD = BC we need to derive the conclusion AD // BC.

Problem 1) Prove the above, using the sequence [1], [2].

1. Connect points B and D, and prove that \( \triangle ABD \equiv \triangle CDB \).

2. From the result of [1], show that \( \angle ADB = \angle CBD \), and prove that AD // BC.

Problem 2) In the diagram for Problem 1, it is also true that AB // DC. Prove this.

From Problems 1 and 2, we see that both pairs of opposite sides of quadrilateral ABCD are parallel. In other words, we can say: "A quadrilateral with both pairs of opposite sides equal is a parallelogram."

This fact is the converse of the property [1] of a parallelogram given on page 124, that the two pairs of opposite sides of a parallelogram are equal.
On the previous page we proved that the converse of property 1 of a parallelogram is also a condition for a quadrilateral to be a parallelogram.

Trigger Let's look at the converse of properties 2 and 3 of a parallelogram on page 124.

Let's try to prove the converse of property 2 of a parallelogram.

**Hint** The hypothesis and conclusion are as follows:

- **Hypothesis** \( \angle A = \angle C, \angle B = \angle D \)
- **Conclusion** \( AB \parallel DC, AD \parallel BC \)

**Proof**

Since the sum of the interior angles of a quadrilateral is 360°,

\[ \angle A + \angle B + \angle C + \angle D = 360^\circ \]

And since \( \angle A = \angle C, \angle B = \angle D \)

\[ \angle A + \angle B + \angle A + \angle B = 360^\circ \]

Therefore

\[ \angle A + \angle B = 180^\circ \]  \( \text{(1)} \)

On the other hand, if we construct the exterior angle at vertex A, \( \angle DAE \)

\[ \angle DAE + \angle DAB = 180^\circ \]  \( \text{(2)} \)

> From (1) and (2), \( \angle DAE = \angle B \)

Since corresponding angles are equal \( AD \parallel BC \)

Similarly \( AB \parallel DC \)

**Note** "Similarly" means we can prove something using a similar procedure.

**Problem 3** If point O is the intersection of the diagonals of quadrilateral ABCD, then,

\( OA = OC, \ OB = OD \)

This is property 3 of a parallelogram. Prove that the converse of this theorem is also true.
Is the quadrilateral ABCD drawn by the following procedures a parallelogram?

1. Draw line segment AD of length 4 cm on line \( l \) on a ruled notepad.
2. Draw line segment BC of length 4 cm on line \( m \) on the notepad.
3. Draw the line segments AB and DC.

In the question above, since the ruled lines on the notepad are parallel, \( AD \parallel BC \) and \( AD = BC \). Does this make the quadrilateral ABCD a parallelogram?

**Problem 4** Prove that quadrilateral ABCD in the question is a parallelogram.

We can collect together conditions for a parallelogram as follows.

---

**Theorem** If any of the following sets of conditions hold for a quadrilateral, it is a parallelogram.

1. Both pairs of opposite sides are parallel. ... Definition
2. Both pairs of opposite sides are of equal length.
3. Both pairs of opposite angles are equal.
4. The diagonals intersect at their midpoints.
5. One pair of opposite sides is equal, and these sides are parallel.

---

**Check 1** Of the following quadrilaterals ABCD, which are always parallelograms?

(a) \( AB = BC, \ AD = DC \)
(b) \( AB = DC, \ AD \parallel BC \)
(c) \( \angle A = \angle C, \ \angle B = \angle D \)
Using the conditions for a parallelogram, we will prove various geometrical facts.

**Example 1**
The diagonals of $\square ABCD$ intersect at point $O$. If $E$ and $F$ are two points on diagonal $BD$ such that $OE = OF$, prove that quadrilateral $AECF$ is a parallelogram.

Which condition for a parallelogram should we use here?

**Proof**
Since the diagonals of a parallelogram intersect at their midpoints,

$$OA = OC \quad \text{(1)}$$

By hypothesis, $OE = OF \quad \text{(2)}$

> From (1) and (2), since the diagonals intersect at

**Check 2**
If the midpoints of sides $AD$ and $BC$ of $\square ABCD$ are $M$ and $N$ respectively, prove that the quadrilateral $MBND$ is a parallelogram.

Monument at Kanazawa Station (Ishikawa Prefecture)
3 Special Cases of the Parallelogram

Let's look at the rectangle, rhombus, and square.

A rectangle is a quadrilateral with four right angles.

A rhombus is a quadrilateral with four equal sides.

Both a rectangle and a rhombus are parallelograms. Can you see why this is so?

A rectangle and a rhombus are both special cases of a parallelogram.

The following facts about their diagonals are also true.

1. The diagonals of a rectangle have the same length.
2. The diagonals of a rhombus are perpendicular.

Problem 1) Draw the diagonals AC and BD of rectangle ABCD. Derive \( \triangle ABC \cong \triangle DCB \).
Then prove that 1 above is true.

Problem 2) Let the diagonals AC and BD of rhombus ABCD intersect at point O. Derive
\( \triangle ABO \cong \triangle ADO \).
Then prove that 2 above is true.
Using the properties of the diagonals of a rectangle, we can prove the following.

The midpoint of the hypotenuse of a right-angled triangle is equidistant from all three vertices of the triangle.

Problem 3) In right-angled triangle ABC, if M is the midpoint of the hypotenuse AC, prove that \( MA = MB = MC \)

A square is a quadrilateral such that all four angles are right angles, and all four sides are the same length. In other words, a square is a quadrilateral that is both a rectangle and a rhombus. Therefore a square has both the properties of a rectangle and those of a rhombus.

Conditions for a rectangle, rhombus, and square

For a parallelogram to be a rectangle, rhombus, or square, what additional conditions have to be satisfied?

Select the conditions from (a) to (d) which fit in each of the numbered (1) to (4) in the diagram below.

- \( \angle A = 90^\circ \)
- \( AB = BC \)
- \( AC = BD \)
- \( AC \perp BD \)

For a parallelogram to be a rectangle, rhombus, or square, what additional conditions have to be satisfied?
Parallel Lines and Area

Using properties of parallel lines we will look at shapes with the same area.

Given a pair of parallel lines. Perpendiculars drawn from two points on one of the lines to the other line are the same length. This is obvious, because opposite sides of a rectangle are the same length.

Example 1

As in the diagram on the right, let's look at the triangles which have common base BC, and apex on a straight line \( \ell \) parallel to BC:

\[ \triangle ABC, \triangle A'BC, \triangle A''BC \]

These triangles have the same base and the same height, and therefore the same area.

In other words, \( \triangle ABC \approx \triangle A'BC = \triangle A''BC \)

Check 1

In \( \square ABCD \) in the diagram on the right, M is the midpoint of the side BC. Find triangles with the same area, and write an expression for this.

Problem 1

If trapezoid ABCD has \( AD \parallel BC \), and O is the point of intersection of its diagonals, then: \( \triangle AOB = \triangle DOC \)

Prove that this is true.

We sometimes write \( \triangle ABC, \triangle A'BC \), and so on, using the name of a figure to represent its area.
Let's think about how to change a polygon without changing its area.

**Example 2** To construct a triangle with the same area as the quadrilateral ABCD in the diagram on the right, we can use the following method.

1. Draw the diagonal AC.
2. Draw line \( \ell \) passing through vertex D, and parallel to AC, and find the point of intersection E with the extension of side BC.
3. Join points A and E, to construct \( \triangle ABE \).

**Problem 2** In the diagram for Example 2, prove that the areas of quadrilateral \( \text{ABCD} = \triangle ABE \).

**Problem 3** In the diagram on the left, a rectangle is divided into two parts, (a) and (b), by a bent line ABC. Draw a straight line passing through A, dividing the rectangle into the same two areas.

**Window on math — Joining midpoints**

Let D and E be the midpoints of sides AB and AC of \( \triangle ABC \).

Explain why the following hold, using area relations.

1. Since \( \triangle DBE = \triangle DCE \),
   \[ DE \parallel BC \]
2. Since \( 2 \triangle DBE = \triangle EBC \),
   \[ 2DE = BC \]
1. **Properties of a parallelogram**

Let M be the midpoint of side CD of quadrilateral ABCD, and let N be the point of intersection of the extension of AD and the straight line BM. If AB = 3 cm and AD = 4 cm, find the lengths of the line segments DM and DN.

2. **Properties of a parallelogram, and conditions for a parallelogram**

In the diagram on the right, quadrilaterals ABCD and EBCF are both parallelograms. Refer to this diagram to answer the following questions.

(1) Prove that AD = EF.

(2) Prove that quadrilateral AEPD is a parallelogram.

3. **Parallel lines and area**

In the diagram on the right, if AB // DC, answer the following questions.

(1) Which triangle has the same area as \( \triangle ABC \)?

(2) In addition to (1), give any other sets of triangles which have the same area.
Place two textbooks as shown in the diagram below, and let A and B be the corners of the textbooks. Taking care not to move A or B, we move a set square touching A and B, and investigate the movement of the corner P.

What is the shape of the path that P moves along?
Problem 1) Draw a number of right-angled triangles ABP with line segment AB as hypotenuse, and see what sort of path the point P plots.

As proved in Problem 3 on page 134, the midpoint of the hypotenuse of a right-angled triangle is equidistant from all three vertices of the triangle. Therefore, if O is the midpoint of the hypotenuse AB of right-angled triangle ABP, we have

\[ OA = OB = OP \]

In other words, point P lies on the circle with line segment AB as diameter.

Problem 2) In the diagram you draw in Problem 1, add the circle with line segment AB as diameter, and check that point P is always on this circle.

Problem 3) As shown in the following diagram, using a set square it is possible to find the center O of a circle. Explain how this works.
Isosceles Triangles and Circles

Two Isosceles Triangles

Let point P be in various positions on the semicircle with center O whose diameter is the line segment AB. What is the size of ∠APB?

In the trigger question, since OA = OB = OP, ∠OPA and ∠OBP are isosceles triangles. In this case, we have proven on page 112 that ∠APB = 90°.

If the base angles of the isosceles triangles are ∠a and ∠b, we can sum up this proof as follows.

The two isosceles triangles OPA and OBP share side OP. .............................. (1)
The three points A, O, and B lie on a straight line. ................................. (2)
Since the sum of the internal angles of Δ ABP is 180°,

\[ 2(∠a + ∠b) = 180° \] ................................. (a)

Therefore, ∠APB = ∠a + ∠b = 90° ................................. (b)

In the diagram for two isosceles triangles as above, if we keep the condition (1), but vary condition (2), we can look at cases where A, O, and B are not on a straight line. In this case, where do the angles \( 2(∠a + ∠b) \) in (a) and ∠a + ∠b in (b) appear?

Try making a tool as shown on the right, and investigate.
Problem 1. In the following diagrams, find the measure of \( \angle a + \angle b \).

In (1) above, \( 2(\angle a + \angle b) \) is equal to difference if we subtract \( \angle AOB \) from 360° (the sum of the interior angles of a quadrilateral). Looking at the diagram on the right, we can see that \( 2(\angle a + \angle b) \) is equal to \( \angle AOB \) measured from the outside of the quadrilateral.

Problem 2. In (2), where is the angle equal to \( 2(\angle a + \angle b) \)? Show this on the diagram on the right.

From what you learned in Problems 1 and 2, we see that given two isosceles triangles sharing side OP, as shown in the diagram on the right:

\[
\angle APB = \frac{1}{2} \angle AOB \\
\text{(1)}
\]

Since OA, OB, and OP are all the same length, we can regard them all as radii of a circle center O. Therefore, we can regard the relation (1) as a relation between \( \angle APB \) formed by any three points A, B, and P on a circle center O, and the central angle \( \angle AOB \).
Inscribed Angles

In the circle center O, if P is a point on the circle excluding the \( \overline{AB} \), then we call \( \angle APB \) an inscribed angle for \( \overline{AB} \). We also call \( \overline{AB} \) the arc intercepted by the inscribed angle \( \angle APB \).

Problem 3 In your notebook, draw a circle center O, and choose two points A and B on the circle, then draw an inscribed angle and central angle for \( \overline{AB} \).

In the circle center O, there are any number of inscribed angles intercepting \( \overline{AB} \), but only one fixed central angle.

### Inscribed angle theorem

**Theorem** The size of an inscribed angle intercepting a particular arc of the circle is constant, and is one-half of the central angle for that arc.

In the diagram above, \( \angle APB = \frac{1}{2} \angle AOB \). We can prove this as follows.

**Proof** Draw diameter PC, then \( \angle APO = \angle a \), \( \angle BPO = \angle b \). Since \( OP = OA \), \( \angle PAO = \angle a \).

Since \( \angle AOC \) is an exterior angle of \( \triangle AOP \), \( \angle AOC = \angle APO + \angle P AO = 2 \angle a \).

Similarly, \( \angle BOC = 2 \angle b \).

Therefore, \( \angle AOB = 2(\angle a + \angle b) \).

Since \( \angle APB = \angle a + \angle b \), \( \angle APB = \frac{1}{2} \angle AOB \).
We can check that in the following cases (a) and (b) it is still true that $\angle APB = \frac{1}{2} \angle AOB$.

(a) \hspace{1cm} (b)

In the diagram on the right, $\angle APB$ is an inscribed angle intercepting $\overline{AB}$, and therefore its size is half of the central angle $\angle AOB$.

Therefore

$$\angle APB = \frac{1}{2} \times 80^\circ = 40^\circ$$

Check 1 In the following diagrams, find the size of $\angle x$.

\[ \begin{align*}
\text{Diagram 1} & \hspace{1cm} \text{Diagram 2} & \hspace{1cm} \text{Diagram 3} \\
\triangle O \hspace{1cm} \triangle O \hspace{1cm} \triangle O
\end{align*} \]

\[ \begin{align*}
\angle O & = 100^\circ \\
\angle O & = 20^\circ \\
\angle O & = 110^\circ
\end{align*} \]

Let's try!

When P is in positions as in diagrams (a) and (b) above, try to prove that the inscribed angle theorem still holds true.
Basic Exercises

Inscribed angle theorem

1. In the following diagrams, find the size of $\angle x$.

1. \[ \angle x \]

2. \[ \angle x \]

3. \[ \angle x \]

---

Window on math — Positioning the video camera

For a choral concert in the school gymnasium, we want to position the video camera so as just to capture the whole stage. In the lower diagram, the current position of the video camera is just right to capture the whole stage, but the locations shaded blue in the diagram are occupied by chairs, and cannot be used. Without changing the zoom setting of the video camera, where should it be located to capture the whole stage?
Chapter Summary Problems A

1. Given an isosceles triangle ABC, with AB = AC. If P is a point on the bisector of the apex \( \angle A \), prove that PB = PC.

2. In the isosceles triangle ABC, AB = BC. The midpoints of AB and AC, are D and E respectively. Let F be the point of intersection of BE and CD.
   1. Prove that BE = CD.
   2. Prove that \( \triangle FBC \) is an isosceles triangle.

3. If two parallelograms ABCD and A'B'C'D each have line segment BD as a diagonal, then all the diagonals, AC, A'C', and BD intersect at a single point. Prove this.

4. As shown in the diagram on the right, from the vertices A and C of \( \square ABCD \), perpendiculatrs are dropped to the diagonal BD, meeting it at E and F respectively. In this case, prove that the quadrilateral AECF is a parallelogram.

5. In the following diagrams, find the size of \( \angle x \).
   (1) \hspace{1cm} (2) \hspace{1cm} (3)

\( \rightarrow p.201 \) Answer p.197 3, p.198 Chapter Summary Problems 145
Chapter Summary Problems • B

1. In \( \triangle ABC \), if the midpoints of sides \( AB \) and \( AC \) are \( D \) and \( E \) respectively, then

\[
DE \parallel BC, \quad DE = \frac{1}{2} BC
\]

Prove this, using the following outline.

1. Prove that if \( F \) is the point on the extension of \( DE \) such that \( EF = DE \), then quadrilateral \( ADCF \) is a parallelogram.
2. Prove that quadrilateral \( DBCF \) is a parallelogram, and derive that \( DC \parallel BC \) and \( DC = \frac{1}{2} BC \).

2. \( C \) is a point on line segment \( AB \), and equilateral triangles \( ACD \) and \( CBE \) are constructed as shown in the diagram on the right, with \( AC \) and \( BC \) as one of each of their sides. In this case, prove that \( AE = DB \).

Let's investigate!

In Question 2 above, if the conditions are changed as follows, what will hold true? Draw a diagram, and write a summary of the results.

- Rotate \( \triangle CBE \) about point \( C \).
- Change the two equilateral triangles to squares.

---

146 5 — Properties of Geometrical Figures  p.202 Answer
Geometrical Figures Maintaining a Constant Angle

Trigger

In the question on page 138, change the point whose movement you are investigating to the 60° corner of a set square, and see what path is traced out by the 60° corner.

As shown in the diagram on the right, if we take many points P such that the angle with respect to the points A and B, \( \angle APB = 60° \), we can expect that point P will lie on a circle with AB as chord.

Problem 1 In your notebook, draw two points A and B, and find ten points P such that \( \angle APB = 60° \). In addition, draw a circle and check if the points are on the same circle.

A circle is a geometrical figure formed by points which meet the condition of being a fixed distance from a point O.

Also, as we found above, we can regard a circle as the set of points P such that the angle formed with respect to two points A and B, \( \angle APB \), is constant.

Problem 1 In Problem 1, change the size of \( \angle APB \) to 45° or 30°, and carry out the same investigation.

\[ \rightarrow p.203 \quad \text{Answer} \]
For a fixed $\overline{AB}$ such that the inscribed angle $\angle ACB = \angle \alpha$, consider whether a point $P$ such that $\angle APB = \angle \alpha$ must lie on the same circle, in the case that point $P$ is on the same side of line segment $AB$ as point $C$.

**Problem 3** Compare the size of $\angle APB$ with the inscribed angle $\angle \alpha$ intercepting the arc $\overline{AB}$, when point $P$ is on the circle, inside the circle or outside the circle center $O$ as shown in the following diagrams.

After investigating as in Problem 3, we find the following.

1. When point $P$ is on the circle center $O$:
   $$\angle APB = \angle \alpha$$

2. When point $P$ is inside the circle center $O$:
   $$\angle APB > \angle \alpha$$

3. When point $P$ is outside the circle center $O$:
   $$\angle APB < \angle \alpha$$

From Problem 3 we see that if point $P$ is on the same side of line segment $AB$ as point $C$, then if $\angle APB = \angle \alpha$ point $P$ lies on the circle center $O$.  

148 5 — Properties of Geometrical Figures
Tracing the Movement of an Inscribed Angle

As shown in the diagram on the right, let's look at inscribed angle $\angle ACB$ that intercepts $AB$ in various positions, and investigate its size.

Trigger Point $P$ moves along the arc $AB$ including point $C$. From point $C$ it approaches point $B$. Mark the angles equal to $\angle C$.

As point $P$ approaches point $B$, we can see that chord $PA$ approaches chord $BA$, and the straight line $PB$ approaches the tangent $BT$ at point $B$.

Problem 4 As point $P$ approaches point $B$, what angle does $\angle P$ approach? Mark this in the diagram on the right.

Problem 5 In Problem 4, move point $P$ more, to go past point $B$ and into the arc on the opposite side. Taking $\angle P$ as the angle made by the straight lines $AP$ and $BP$, where does this angle appear in the diagram on the right. Mark the angle in the diagram.
In the diagram for Problem 5, we can view all of the vertices of quadrilateral APBC as lying on a circle. When a quadrilateral has all four vertices lying on a circle in this way, we say it is a cyclic quadrilateral, and is inscribed in the circle, while the circle is the circumcircle of the quadrilateral.

**Problem 6** The diagram on the right shows Problem 5 on the previous page, with E a point on the extension of BP. Prove the following two statements.

1. \( \angle ACB = \angle APE \)
2. \( \angle ACB + \angle APB = 180^\circ \)

Statement 2 in Problem 6 shows the following.

Opposite angles of a cyclic quadrilateral add up to 180°.

From Problem 4 on the previous page, we can expect that as in the diagram on the right, \( \angle ABT \), the angle between a tangent BT to the circle and the chord AB is equal to \( \angle ACB \), the inscribed angle that intercepts AB.

**Problem 7** In the diagram on the right, BT is a tangent to the circle center O, with its point of contact at B. Prove that \( \angle ABT = \angle ACB \).

From what you found in Problem 7, we can see the following.

The angle between a chord and a tangent passing through the same point on a circle is equal to the inscribed angle in the circle that intercepts the chord.
Converting a quadrilateral
Cut the quadrilateral on the right into four pieces, cutting along straight lines, and rearrange the pieces to form a parallelogram.

64 = 65?
Cut the square shown in the diagram below along the thick blue lines, and rearrange the pieces to form the rectangle on the right. Why does the area increase?

8 \times 8 = 64
5 \times 13 = 65

You can cut out the version printed on p. 209.
Chapter 6 Probability

There are six cards, each of which has a single number from 1 to 3 written on it, as follows: 1, 2, 2, 3, 3, 3. Shuffle these cards well, and draw one, to investigate how the numbers come out.

Cut out the cards from page 209, and try an actual experiment, entering the numbers in the table below.

Note that after drawing a card, you must return it to the pack for the next round.

<table>
<thead>
<tr>
<th>Number of try</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number on card</td>
<td>11</td>
<td>12</td>
<td>13</td>
<td>14</td>
<td>15</td>
<td>16</td>
<td>17</td>
<td>18</td>
<td>19</td>
<td>20</td>
</tr>
<tr>
<td></td>
<td>21</td>
<td>22</td>
<td>23</td>
<td>24</td>
<td>25</td>
<td>26</td>
<td>27</td>
<td>28</td>
<td>29</td>
<td>30</td>
</tr>
</tbody>
</table>

Which number appears most often?
Do some events occur more often than others?

In each of the following questions there are two events a and b. Which of the two do you think will occur more often?

1. When you throw a die:
   - a. the result is a 1
   - b. the result is an even number

2. If you toss a 100-yen coin
   - a. it lands heads up (cherry blossom side)
   - b. it lands tails up ("100" side)

3. Drawing card from a pack of 52 playing cards (no jokers)
   - a. the card is a heart
   - b. the card is a court card (KQJ)

Let's try!

With the set of six cards described in the trigger question on the previous page, draw a card, then replace this card and draw a card once more.

In this case, which of the following, a or b do you think is more likely to happen?

- a. Both times the card is a 3.
- b. You draw one 2 and one 3.
1 Thinking About Probability

Let's think about how to represent how likely an event is, using numbers.

In an experiment of throwing a regular die, all sides will land uppermost equally often. Therefore, we can consider that a 1 will appear about once in every six times. In other words, if we throw a die a large number of times, the proportion of times the 1 lands uppermost will be close to \( \frac{1}{6} \).

\[
\text{(Proportion of 1 occurring)} = \frac{\text{(Number of times 1 occurred)}}{\text{(Total number of throws)}}
\]

The table on the right shows the result of an experiment to find out how many times 1 occurred after throwing a die.

<table>
<thead>
<tr>
<th>Number of throws</th>
<th>Number of times 1 occurred</th>
<th>Proportion of 1 occurring</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>7</td>
<td>0.14</td>
</tr>
<tr>
<td>100</td>
<td>13</td>
<td>0.13</td>
</tr>
<tr>
<td>200</td>
<td>32</td>
<td>0.16</td>
</tr>
<tr>
<td>400</td>
<td>70</td>
<td>0.175</td>
</tr>
<tr>
<td>600</td>
<td>89</td>
<td>0.148</td>
</tr>
<tr>
<td>800</td>
<td>125</td>
<td>0.156</td>
</tr>
<tr>
<td>1000</td>
<td>165</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>202</td>
<td></td>
</tr>
<tr>
<td>1400</td>
<td>239</td>
<td></td>
</tr>
<tr>
<td>1600</td>
<td>269</td>
<td></td>
</tr>
<tr>
<td>1800</td>
<td>299</td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>334</td>
<td></td>
</tr>
</tbody>
</table>

We can show the results of this experiment using a line graph shown on the next page.
Problem 2) From the results of Problem 1, complete the graph above. What can you understand from the graph?

Experiments or observations such as "throwing a die and getting a 1" are affected by chances. The possible result of a single trial in an experiment is called an outcome. An outcome or group of outcomes is called an event. The measure of the likelihood that an event will occur is called the probability of the event.

For example, when we throw a die we expect that the result will be a 1 just once in six times, and therefore the probability is $\frac{1}{6}$.

Thus to say that the probability of $P$ means that if we repeat the experiment a large number of times, the proportion of times when the event occurs will approach $P$.

Check 1 In an experiment of tossing a 10-yen coin many times, what value do you expect the proportion of heads to approach? Give the probability of heads and the probability of tails.

Problem 3) In an experiment to throw a die six times, Albert made the following prediction. Is this prediction correct?

Since the probability of a 1 is $\frac{1}{6}$, a 1 will always appear exactly once in six throws.
Let's look at probability, using data from a number of sources.

Let's examine the table below.

**Problem 4** The table on the right shows the number of births in Japan in the years from 1991 to 2002. It also shows the number and proportion of baby girls born each year.

From the table on the right, can we say that the probability of a baby girl being born is $\frac{1}{2}$?

Well, since a baby is either a boy or a girl, I suppose the probability of a girl being born is $\frac{1}{2}$.

The probability in Problem 4 is not like the probability found in Check 1, but is based on the result of a large number of actual observations.

**Check 2** It is said that October 10 is often fine weather. Looking at the weather records for Fukuoka City for the 100 years from 1903 to 2002, it was found that this day was fine weather in 64 years. Find the probability of fine weather on October 10 in Fukuoka City.
In a soccer match, to decide which team kicks off there is a coin toss. Let's think about why we use a coin.

When we toss a coin, we have equal expectations of it landing a head or a tail. In such cases we say that the outcomes are equally likely. In such cases we can calculate probabilities.

**Example 1**

Let's find out the probability of getting an odd number in an experiment of throwing a die.

As shown in the diagram on the right, there is a total of six possible results. These results are all equally likely outcomes. Of these six results, there are three outcomes with an odd number, so:

\[
\text{Probability of an odd number} = \frac{3}{6} = \frac{1}{2}
\]

**Check 1**

Answer the following questions to find the probability of obtaining an even number in throwing a die.

1. How many outcomes are there in which the result is an even number?
2. What is the probability of the result being an even number?
In general the following result holds.


Finding a probability

Suppose that in an experiment or observation there are a total of \( n \) possible outcomes or results and each of these outcomes is equally likely to happen. Then if there are \( a \) ways for event \( A \) to occur, the probability \( P \) of \( A \) occurring is given by the following.

\[
P = \frac{a}{n}
\]

Example 1

I have ten cards, each marked with a different number from 1 to 10. I shuffle these cards well, then draw a card. Find the probability that the card drawn is a multiple of 3.

Use the following procedure to approach this problem.

1. There are ten possible outcomes, and these are all equally likely.
2. There are three cases in which the number on the card is a multiple of 3: 3, 6, and 9.
3. Therefore, the required probability is \( \frac{3}{10} \).

Check

If a card is drawn from a 52-card pack of playing cards (no jokers), find the probability that the card is a heart, using the following procedure.

1. How many outcomes are there in total?
2. Can we say that the events in (1) are all equally likely?
3. How many outcomes are there in which the card is a heart?
4. Find the probability that the card drawn is a heart.
Let's investigate the range of probabilities.

**Trigger**

Three bags, A, B, and C, each contain five balls, as follows. If we take one ball out, for each bag we will calculate the probability that it is a blue ball.

Bag A: 3 blue, 2 white balls  
Bag B: 5 blue balls  
Bag C: 5 white balls

The range of a probability is from 0 to 1. This is $0 \leq p \leq 1$.

A probability of 1 means the event will certainly occur.

A probability of zero means the event will never occur.

**Problem**

Insert the symbol for one of the suits, (hearts, diamonds, clubs, and spade) in the $\Box$ in the following statement to make a problem for which the required probability is 1 or 0.

If a card is drawn from the set of 13 hearts, find the probability that the suit of the card is $\Box$.

---

**Window on math — The beginnings of probability**

The mathematician Pascal was asked the following question by his friend Méré: "Two friends A and B are playing dice. The first player to win three games takes the stakes, but if the game has to be abandoned after one player has won twice and the other once, how should the stakes be divided?"

It was this problem that led to the development of probability theory in the 17th century.
We will list all possible outcomes in a table or diagram, then find the desired probability.

Example 3
I toss a 100-yen coin twice. What is the probability that heads come up in both tosses?

Hint
Try writing the outcomes in a table as on the right.

Solution
We can write the outcome of heads first time and tails second times as (Heads, Tails).

There are four outcomes with the following results.
(Heads, Heads), (Heads, Tails), (Tails, Heads), and (Tails, Tails)

We can expect that each of these is equally likely.

Of these only one represents heads both times.

Therefore the probability is \( \frac{1}{4} \).

Check 3
In Example 3, find the probability of heads come up the first time and tails come up the second time.

Problem 2
Three people, A, B, and C have decided to play the following game.

Two coins will be tossed. If both land heads up A wins, if one is heads and one is tails B wins, and if both land tails up C wins.

A gives the following argument about this game.

There are three possible outcomes:

- two heads, one each heads and tails, and two tails

Therefore the probability of each of the three players winning is \( \frac{1}{3} \).

Is this argument correct?
To list all of the possible outcomes, we can make a table as shown in Example 3 on the previous page. We can also use a diagram as shown on the right. For instance, in Example 3, we can draw a diagram as on the right, to show all of the outcomes. H represents the head and T represents the tail.

This kind of diagram is called a **tree diagram**.

Let's find more probabilities. Using a tree diagram.

Check 4 A and B play rock-paper-scissors. For a single round, answer the following questions.

1. If we write R for rock, P for paper, and S for scissors, complete the diagram on the right.
2. Find the probability that A wins.

If A play rock, there are three ways for B to play: rock, paper, and scissors.

Problem 3 In the above problem, find the probability of a draw.
Example 4) Two dice, one large, one small, are thrown. Find the probability that the total on the two dice is 5.

Hint > We will write the outcome in which the small die is 2 and the large die is 3 as (2, 3). All other possible outcomes are listed in the following table.

<table>
<thead>
<tr>
<th>Large</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>Small</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(1,1)</td>
<td>(1,2)</td>
<td>(1,3)</td>
<td>(1,4)</td>
<td>(1,5)</td>
<td>(1,6)</td>
</tr>
<tr>
<td>2</td>
<td>(2,1)</td>
<td>(2,2)</td>
<td>(2,3)</td>
<td>(2,4)</td>
<td>(2,5)</td>
<td>(2,6)</td>
</tr>
<tr>
<td>3</td>
<td>(3,1)</td>
<td>(3,2)</td>
<td>(3,3)</td>
<td>(3,4)</td>
<td>(3,5)</td>
<td>(3,6)</td>
</tr>
<tr>
<td>4</td>
<td>(4,1)</td>
<td>(4,2)</td>
<td>(4,3)</td>
<td>(4,4)</td>
<td>(4,5)</td>
<td>(4,6)</td>
</tr>
<tr>
<td>5</td>
<td>(5,1)</td>
<td>(5,2)</td>
<td>(5,3)</td>
<td>(5,4)</td>
<td>(5,5)</td>
<td>(5,6)</td>
</tr>
<tr>
<td>6</td>
<td>(6,1)</td>
<td>(6,2)</td>
<td>(6,3)</td>
<td>(6,4)</td>
<td>(6,5)</td>
<td>(6,6)</td>
</tr>
</tbody>
</table>

Solution > There are a total of 36 possible outcomes, and we can expect these all to be equally likely. Of these the following four outcomes give a total of 5:

(1, 4), (2, 3), (3, 2), (4, 1)

Therefore the required probability is

\[
\frac{4}{36} = \frac{1}{9}
\]

Check 5 If large and small dice are thrown, find the probability that the total is 6.

Problem 4) When large and small dice are thrown, answer the following questions.

1) What total on the two dice has the highest probability of occurring?

2) Find the probability that the number on the large die is greater than the number on the smaller die.
Various Probabilities

When drawing lots, are you more likely to win by drawing first or by drawing second?

Drawing first looks better to me...

Ah, but they do say "Last but not the least"...

Example 1

There are five paper slips, and three of them have a red "Win" mark. If A and B draw one each in turn, which of A and B is likely to win?

Hint

Number the papers, with the winning ones 1, 2, and 3, and the losing ones 4 and 5. If we draw a tree diagram of A and B's draws, there are 20 outcomes, as shown on the right. We can regard all of these as equally likely to occur.

Problem 1

In Example 1, find the probabilities of A and B winning. Which of A and B is more likely to win?

Let's try!

In Example 1, if we vary the number of paper slips, the number of winning slips, or the number of people drawing, see if the probability of winning is affected by the order of drawing.

1 — Probability 163
Example 2) Four students, A, B, C, and D have to select a pair among them for duties by drawing lots. In this case, find the probability that D is selected.

Hint For example, whether A and B are selected, or B and A are selected makes no difference. Noting this, we can list all of the combinations for duties as follows.

\{A, B\}, \{A, C\}, \{A, D\}

\{B, C\}, \{B, D\}

\{C, D\}

And we see there are six combinations, which we can assume are equally likely to occur.

Solution Of the total of six outcomes, there are three combinations including student D. Therefore the probability required is:

\[
\frac{3}{6} = \frac{1}{2}
\]

Answer \(\frac{1}{2}\)

Problem 2) Three members, A, B, and C, of the table tennis club draw lots to choose two players for a doubles team. Find the probability that A is included in the team.
Basic Exercises

Probability

1. The following table shows the results of throwing a bottle cap.

<table>
<thead>
<tr>
<th>Number of times thrown</th>
<th>100</th>
<th>500</th>
<th>1000</th>
<th>1500</th>
<th>2000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of &quot;Heads&quot;</td>
<td>39</td>
<td>179</td>
<td>368</td>
<td>564</td>
<td>742</td>
</tr>
<tr>
<td>&quot;Heads&quot;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>&quot;Tails&quot;</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(1) Which of "Heads" and "Tails" appears to be more likely to occur?

(2) What value does the proportion show in the table approach? Find the value to two decimal places.

Procedures for finding probability

2. A bag contains three red balls, two blue balls, and four white balls. Answer the following questions to find out the probability of getting a blue ball from the bag without looking.

(1) How many possible outcomes are there in total?

(2) Can we regard the outcomes in (1) as all equally likely to occur?

(3) How many outcomes result in a blue ball being drawn?

(4) Find the probability of drawing a blue ball.

Tree diagrams and probability

3. Four cards are marked with the numbers 1, 2, 3, and 4. We shuffle the cards well, then draw two cards, and lay them in the order drawn to form a two-digit integer.

(1) Draw a tree diagram, to list all the two-digit numbers that can be formed.

(2) Find the probability that the number formed is at least 32.

→ p. 174 See "Finding the Number of Cases" (development) 1 — Probability 165
There are three girls, Alison, Brenda, and Celia, and two boys, David and Earnest in a tennis club. A girl and a boy are chosen by drawing lots to form a pair for a mixed doubles tennis match.

1. How many possible pairs are there in total?
2. Find the probability that Alison and Earnest are the pair.

A bag contains one red ball, one white ball, and one blue ball. I draw the balls from the bag one at a time, and lay them in a row in the order drawn.

1. How many different ways can the balls be drawn?
2. Find the probability that the red and white balls are adjacent.

Six cards are numbered 1 to 6. I shuffle these cards, then draw two cards in succession, and lay the cards in the order I drew them, to make a two-digit number. In this case, find the probability that the number formed is a multiple of 3.

When the integer of tens digit and the units digit is a multiple of 3, the integer is always a multiple of 3.

Enter a word in the blank in the following question, so that the probability to be found is equal to \( \frac{1}{2} \).

If a die is thrown, find the probability that the number given is
1. Two dice, one large, one small, are thrown; let \( x \) be the number on the larger die and \( y \) be the number on the smaller die. Find the probability that \( 2x + y = 8 \).

2. There are six cards, each of which has a single number from 1 to 3 written on it, as follows: 1, 2, 2, 3, 3, 3. These cards are shuffled well, and one is drawn, then replaced, then another is drawn. Which is more likely to happen: that a 3 is drawn both times, or that a 2 and a 3 are drawn once each?

3. The diagram below shows a regular hexagon ABCDEF. The bag contains five cards, labeled B, C, D, E, and F. We draw two cards from the bag, and join the vertices of the letters drawn and the vertex A. Thus drawing a triangle. What is the probability that this is a right-angled triangle?

\[ \text{Let's investigate!} \]

Find various probabilities, by carrying out experiments.

- Find the probability of a bottle cap landing "heads" (outside up) or "tails" (inside up) or a thumbtack landing pointing up or down.
- Try carrying out the experiment on page 193.
Probability of an Event Not Occurring

If the probability of winning when drawing lots is \( \frac{1}{4} \), what is the chance of not winning?

Let's look at the relation between an event occurring and the same event not occurring.

Example 1) When a die is thrown, find the probability that the result is not a 1.

The ways for a die to land are:

\[
\begin{array}{cccccc}
\Box & \Box & \Box & \Box & \Box & \Box \\
\end{array}
\]

And all of these outcomes are equally likely.

For the event "the result is not a 1" to occur, there are five possible outcomes:

\[
\begin{array}{cccccc}
\Box & \Box & \Box & \Box & \Box & \Box \\
\end{array}
\]

Therefore the probability that the result is not a 1 is \( \frac{5}{6} \).

As we saw above, the probability that the result is not a 1 is \( \frac{5}{6} \). But the probability that the result is a 1 is \( \frac{1}{6} \). From these facts we see that the following relation holds:

\[
(\text{Probability of 1}) + (\text{Probability of not 1}) = 1
\]

In general, for an event A, the following holds:

\[
(\text{Probability of A occurring}) + (\text{Probability of A not occurring}) = 1
\]

Therefore, the following relation holds:

\[
(\text{Probability of A not occurring}) = 1 - (\text{Probability of A occurring})
\]

Problem 1) If three 10-yen coins are tossed, find the probability that at least one coin lands tails up.

"At least one coin is tails" is the same as "All three are heads" not occurring.
Appendix

- Further Topics
  - Equations with Three Unknowns (development) 170
  - Finding the Number of Cases (development) 174
  - Making Predictions from Data (development) 178
  - When Are the Clock Hands Aligned? 182

- Private Study
  - How Many Turns on the Toilet Roll? 186
  - Traditional Patterns with Tiled Figures 188
  - Reading Braille 190
  - Let's Make a Beam Balance 192
  - Experimenting with Dice for Random Numbers 193

- Supplementary Problems 194
Equations with Three Unknowns

In the diagram on the right, each number in a blue box is the sum of the two values in pink circles on either side. Let's find the values of \( x, y, \) and \( z. \)

We will investigate the relations among \( x, y, \) and \( z, \) and find their values.

Problem 1) Albert compares the sum of \( x \) and \( y \) with the sum of \( x \) and \( z, \) and realizes that \( y \) is 2 more than \( z. \)

How did he get this?

From Problem 1 and the fact that the sum of \( y \) and \( z \) is -4, we get the following simultaneous equations.

\[
\begin{align*}
y - z &= 2 \\
y + z &= -4
\end{align*}
\]

Problem 2) Solve the above simultaneous equations, and find the values of \( y \) and \( z. \) From the values of \( y \) and \( z, \) find the value of \( x. \)

Albert found a relation between the values represented by \( y \) and \( z, \) and derived simultaneous equations.

In this problem, there are three values to be found; can we find the values using \( x, y, \) and \( z, \) by representing the relations among them as equations?

Answer
If we write the conditions of the problem as equations using $x$, $y$, and $z$, we get the three following simultaneous equations:

\begin{align*}
  x + y &= 3 \quad \text{(1)} \\
  y + z &= -4 \quad \text{(2)} \\
  x + z &= 1 \quad \text{(3)}
\end{align*}

If we look at Albert's solution using the equations, we can find the values of $x$, $y$, and $z$ satisfying the above three equations, by the following method.

1. Eliminate $x$ from equations (1) and (3), to derive the equation $y - z = 2$.
2. Solve $y - z = 2$ and equation (2) as simultaneous equations, to find the values of $y$ and $z$.
3. From the values of $y$ and $z$, find the value of $x$.

**Problem 3.** In the above procedure, answer the following questions.

1. In (1), how can we eliminate $x$ from the equations (1) and (3)?
2. In (3), how can we find the value of $x$ from the values of $y$ and $z$?

In Chapter 2 you learned about simultaneous equations with two unknowns, but here we are dealing with three unknowns.

Generally, we talk of a linear equation in three unknowns, and a combination of such equations we call a system of simultaneous linear equations in three unknowns.

We can say the equations we used here are a system of simultaneous linear equations in three unknowns, with coefficients 0 or 1, as shown on the right.
Let's solve a set of simultaneous linear equations in three unknowns, where the coefficients are not all 0 or 1.

**Example 1** Solve the following simultaneous equations.

\[
\begin{align*}
2x + y + 4z &= 5 \quad \text{(1)} \\
-x - y + 2z &= -2 \quad \text{(2)} \\
2x - y - z &= 7 \quad \text{(3)}
\end{align*}
\]

**Hint** We can eliminate \( z \) as follows.

\[
\begin{align*}
(1) \\
(2) \times 2 \\
(3) \times 2
\end{align*}
\]

\[
\begin{align*}
2x + y + 4z &= 5 \\
-2x - 2y + 4z &= 4 \\
4x + 3y &= 9 \quad \text{(4)}
\end{align*}
\]

\[
\begin{align*}
-x - y + 2z &= -2 \\
4x - 2y - 2z &= 14 \\
3x - 3y &= 12 \quad \text{(5)}
\end{align*}
\]

**Problem 4** Eliminate \( y \) from (4) and (5), to find the value of \( x \). Using this value, then find the values of \( y \) and \( z \).

Thus, to solve simultaneous linear equations in three unknowns, first eliminate one unknown, and solve the resulting equations in two unknowns.

**Problem 5** Solve the following sets of simultaneous equations.

\[
\begin{align*}
\begin{align*}
x + 5y - 2z &= 7 \\
x - 4y + z &= -5 \\
7x - 3y - z &= 0
\end{align*} & \begin{align*}
2x + y + 3z &= 12 \\
3x + 2y - z &= 1 \\
x - 2y + 4z &= 16
\end{align*}
\end{align*}
\]

**Problem 6** There are three natural numbers, \( a, b, \) and \( c \). If we divide \( c \) by \( b \), the quotient is 7 and the remainder 4.

When we divide the sum of \( b \) and \( c \) by \( a \), the quotient is 5 and the remainder 4. The sum of \( a, b, \) and \( c \) is 100. What are the three numbers?
In the \textcolor{black}{\textit{text}} on page 170, we can also solve using the following approach.

Each number in a blue \textcolor{red}{\textit{box}} is the sum of the numbers in the \textcolor{blue}{\textit{circle}} on either side. So the sum of the numbers in the three \textcolor{red}{\textit{boxes}} is twice the sum of the numbers in the \textcolor{blue}{\textit{circle}}. Therefore, the sum of the numbers in the \textcolor{blue}{\textit{circle}} is \((3 - 4 + 1) / 2 = 0.\)

**Problem 7** Using the above approach, how can we find the values of the numbers in the \textcolor{blue}{\textit{circle}}?

If we write the above approach in equations, we get the following.

\[
\begin{align*}
    x + y &= 3 \\ 
    y + z &= -4 \\ 
    x + z &= 1
\end{align*}
\]

From \((1) + (2) + (3)\) we get:

\[
2x + 2y + 2z = 0 \\
\Rightarrow x + y + z = 0 \quad (4)
\]

From \((4) - (1)\) \quad z = -3

Similarly \(x = 4, y = -1\)

\[
x = 4, y = -1, z = -3
\]

**Problem 8** In the diagram \textcolor{black}{\textit{on the right}}, find the values of \(x, y,\) and \(z.\)

Try making more problems like this.
Finding the Number of Cases

Four soccer teams play each other "home-and-away". How many matches is this in total?

Try to think of a neat way of counting all the possibilities, being careful not to double-count, nor to miss some.

In the table, if the teams are a, b, c, and d, then if we represent team a playing at home against team b playing away as ab, all of the matches are as listed in the table on the right.

<table>
<thead>
<tr>
<th></th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>×</td>
<td>ab</td>
<td>ac</td>
<td>ad</td>
</tr>
<tr>
<td>b</td>
<td>ba</td>
<td>×</td>
<td>bc</td>
<td>bd</td>
</tr>
<tr>
<td>c</td>
<td>ea</td>
<td>eb</td>
<td>×</td>
<td>cd</td>
</tr>
<tr>
<td>d</td>
<td>da</td>
<td>db</td>
<td>dc</td>
<td>×</td>
</tr>
</tbody>
</table>

Problem 1) How do we approach the problem to calculate the total number of matches? Use the table above as reference.

Problem 2) Suppose that instead of home-and-away, each pair of teams plays just one match, then how can we look at the above table to find the total number of matches? Using this approach, find the total number.

Problem 3) Find the total number of brown dots in the diagram on the right, using the approach of Problem 2.

"Home and away" means each team plays each other team twice: once at its home ground, and once at the opponent's ground.
To find the total number of matches in a home-and-away tournament, for example we can use the following approach.

1. There are four cases for the home team: \( a, b, c, \) and \( d \)

2. For each team playing at home, the other three teams will each be playing an away match.

3. Therefore the total number is \( 4 \times 3 = 12 \) matches.

This approach is the same as asking how many ways there are of selecting two of the four letters \( a, b, c, \) and \( d, \) and putting them in a sequence.

Example 1

A bag contains one each of red, blue, black, and white balls. From this bag, I draw three balls in turn, and place them in sequence. How many different sequences are there of the balls drawn from the bag?

**Hint**

For the first ball, there are four possibilities.

For the second ball, there are three possibilities for each first ball.

For the third ball, there are two possibilities for each second ball.

**Solution**

The number of sequences is

\[
4 \times 3 \times 2 = 24
\]

Answer 24 sequences

Problem 4

Select two from the five numerals 1, 2, 3, 4, and 5, and arrange them to make a two-digit number.

How many two-digit numbers can we make?
In an all-play-all competition, where each pair of teams plays one match only, we do not distinguish \( ab \) for \( a \) as the home team against team \( b \) from \( ba \) for \( b \) as the home team against team \( a \), and are only concerned with the combinations of matched teams.

In a four-team all-play-all, we can find the number of matched teams to play once each, by taking the four points \( a, b, c, \) and \( d \) in the diagram on the right, and counting the number of line segments joining any two of the four points.

1. Since the points that can be joined to \( a \) are the three points other than \( a \), we can draw three lines from \( a \).

2. Similarly, there are the same number of lines joining \( b, c, \) and \( d \) to other points, so the total number of lines is \( 3 \times 4 \), that is 12 lines.

3. However, when drawing a line from \( a \) to \( b \), we have drawn the same line as when drawing from \( b \) to \( a \), and therefore we have counted each line twice.

4. Therefore we can find the number of lines joining two of four points as \( 12 \div 2 \), that is, 6 lines.

In a four-team all-play-all, the combinations for a single match each is as follows.

\[ ab, ac, ad, bc, bd, cd \]

**Problem 5** Two people are to be chosen from a group of ten. How many ways can they be chosen?
Problem 6) Find the number of diagonals of a decagon (10 sides).

Problem 7) A bag contains one each of red, blue, black, and white balls. I take three balls simultaneously from the bag. How many ways are there of doing this?

Problem 8) Four people are being chosen from a group of six for a relay team. How many ways are there of doing this with the following selection methods:

1) Selecting team members with a sequence for running.
2) Selecting the members without deciding any sequence.

To find the total number of cases when it cannot be calculated immediately, draw a tree diagram or table, and count the cases carefully in sequence.

Problem 9) How many combinations of coins can I use if I buy a can of soft drink for 120 yen from a vending machine? The only coins that can be used are 10 yen, 50 yen, and 100 yen, and there should be no change.

Further Topics 177
Making Predictions from Data

The following table shows the average air temperature (in °C) in Hirosaki for March every year from 1983 to 2004. The dates when the Yoshino ornamental cherry trees were in full bloom are also shown. Let's analyze the relationship of these two data.

<table>
<thead>
<tr>
<th>Year</th>
<th>Average March temperature</th>
<th>Full-bloom date</th>
<th>Year</th>
<th>Average March temperature</th>
<th>Full-bloom date</th>
</tr>
</thead>
<tbody>
<tr>
<td>1984</td>
<td>-1.3</td>
<td>5 6</td>
<td>1995</td>
<td>2.0</td>
<td>4 21</td>
</tr>
<tr>
<td>1985</td>
<td>0.6</td>
<td>4 26</td>
<td>1996</td>
<td>1.2</td>
<td>4 27</td>
</tr>
<tr>
<td>1986</td>
<td>1.2</td>
<td>4 26</td>
<td>1997</td>
<td>2.3</td>
<td>4 22</td>
</tr>
<tr>
<td>1987</td>
<td>1.3</td>
<td>4 23</td>
<td>1998</td>
<td>3.3</td>
<td>4 14</td>
</tr>
<tr>
<td>1988</td>
<td>0.9</td>
<td>4 27</td>
<td>1999</td>
<td>1.3</td>
<td>4 22</td>
</tr>
<tr>
<td>1989</td>
<td>4.2</td>
<td>4 14</td>
<td>2000</td>
<td>1.2</td>
<td>4 26</td>
</tr>
<tr>
<td>1990</td>
<td>3.8</td>
<td>4 13</td>
<td>2001</td>
<td>1.8</td>
<td>4 19</td>
</tr>
<tr>
<td>1991</td>
<td>1.7</td>
<td>4 22</td>
<td>2002</td>
<td>4.2</td>
<td>4 14</td>
</tr>
<tr>
<td>1992</td>
<td>2.5</td>
<td>4 22</td>
<td>2003</td>
<td>1.8</td>
<td>4 19</td>
</tr>
<tr>
<td>1993</td>
<td>2.6</td>
<td>4 23</td>
<td>2004</td>
<td>2.8</td>
<td>4 16</td>
</tr>
<tr>
<td>1994</td>
<td>1.1</td>
<td>4 22</td>
<td>2005</td>
<td>1.0</td>
<td>4 28</td>
</tr>
</tbody>
</table>

From the data for 1984 and 2002, what relationship can we say there is between the average temperature and the cherry blossom full-bloom date?

It is hard to see the relation just from looking at the table, so let's make a graph.

**Problem**

Let the average March temperature be \( t \) °C, and the cherry blossom full-bloom date be the \( n \)th day of April, then mark the points in the graph on the following page with the pairs of \( t \) and \( n \) coordinates.
The graph above is called a correlation diagram. It is also called a scatter plot diagram. A correlation diagram is a graph showing the relation between two quantities. Based on the correlation diagram you drew in Problem 1, we will draw a straight line passing as close as possible through the plotted points. We call this line the line of best fit. We will use the line of best fit as a guideline to find the full-bloom date for 2004.

**Problem 2**
In the correlation diagram for Problem 1, draw a straight line, passing as close as possible through the middle of the points. Find an equation for this straight line. Using this equation, find the predicted full-bloom date for 2004.

**Problem 3**
In March 1980, the average temperature in Hirosaki was 0.8 °C. Find a prediction for the cherry blossom full-bloom date this year, using the equation you found in Problem 2.

In this way, if we can derive a function best fitting the data in a correlation diagram, we can investigate trends in the data and make various predictions. Graphing calculators and computer spreadsheet software provide functions for deriving equations in this way.
Let's look at correlation for a different example.

Akiko, who suffers from hay fever every year, was investigating hay fever, and found a graph like that on the right. This graph shows the yearly data in Tokyo for the pollen counts of cryptomeria and hinoki cypress, and the total sunlight in July of the previous year.

What can we see from this graph?

To find the relation between pollen count and sunlight, we'll make a correlation diagram.

Problem 4: Answer the following questions about the relation between pollen count and sunlight.

1. Based on the data in the following table, draw a correlation diagram for the relation between the two quantities.

<table>
<thead>
<tr>
<th>Year</th>
<th>Sunlight in previous July</th>
<th>Pollen count</th>
<th>Year</th>
<th>Sunlight in previous July</th>
<th>Pollen count</th>
</tr>
</thead>
<tbody>
<tr>
<td>1981</td>
<td>11.5</td>
<td>534</td>
<td>1991</td>
<td>15.8</td>
<td>2865</td>
</tr>
<tr>
<td>1982</td>
<td>17.6</td>
<td>4567</td>
<td>1992</td>
<td>13.5</td>
<td>1001</td>
</tr>
<tr>
<td>1983</td>
<td>11.6</td>
<td>350</td>
<td>1993</td>
<td>14.6</td>
<td>3751</td>
</tr>
<tr>
<td>1984</td>
<td>13.1</td>
<td>1669</td>
<td>1994</td>
<td>9.6</td>
<td>277</td>
</tr>
<tr>
<td>1985</td>
<td>15.1</td>
<td>2026</td>
<td>1995</td>
<td>17.6</td>
<td>7588</td>
</tr>
<tr>
<td>1986</td>
<td>16.0</td>
<td>1603</td>
<td>1996</td>
<td>14.6</td>
<td>972</td>
</tr>
<tr>
<td>1987</td>
<td>12.0</td>
<td>496</td>
<td>1997</td>
<td>16.2</td>
<td>2808</td>
</tr>
<tr>
<td>1988</td>
<td>15.9</td>
<td>2532</td>
<td>1998</td>
<td>16.1</td>
<td>2110</td>
</tr>
<tr>
<td>1989</td>
<td>11.3</td>
<td>412</td>
<td>1999</td>
<td>12.5</td>
<td>673</td>
</tr>
<tr>
<td>1990</td>
<td>13.7</td>
<td>1842</td>
<td>2000</td>
<td>17.7</td>
<td>4804</td>
</tr>
</tbody>
</table>

2. From the correlation diagram, what can we say about the relation between the two quantities?

* The count of pollen grains observed in the season over a 1 cm² area.

** The energy from the sun converted to heat, in units of MJ/m².

180
Let's collect data from an experiment, and make predictions from it.

A baseball or soccer crowd sometimes makes a giant "wave" by everyone standing with their hands high in turn.

In your class, try making a wave with different numbers of people, and from the data collected predict how long one cycle of a wave lasts.

1. Make a wave, standing in a circle, and measure the time for the wave to go one cycle, as you increase the number of people from 5, 10, 15, and so on.

2. With the number of people as $x$ and the time for one cycle as $y$ seconds, draw a correlation diagram for the collected data.

3. Draw the line of best fit in the correlation diagram. And find an equation for this straight line.

4. Using the equation you found in 3, find the predicted time for one cycle of a wave with all of the students in the class.

Problem 5 Make a wave with all the students, and compare with the time predicted in 4.

Problem 6 Predict how long one cycle would take for a wave created by all students in the school.
When Are the Clock Hands Aligned?

Try solving the following problem by different approaches.

The hour hand and the minute hand of a clock are aligned at 12 o'clock. Twelve hours later they are aligned again in the same way. In this interval, the hands have been aligned a number of times; find the times when this occurs.

---

**Approach using equations**

**Trigger** Through how many degrees do the hour and minute hands turn per minute?

The hands are aligned when their positions are the same angle from the 12-o'clock position on the clock face. From this we can construct equations.

**Problem** Answer the following question if the hands are aligned at $x$ minutes after 1 o'clock.

1. When the hands are aligned, write an expression using $x$ for the position of each hand as an angle from 12 o'clock on the clock face.
2. From the angle relation, write an equation.
3. Solve the equation you wrote in (2), to find the number of minutes after 1 o'clock when the hands are aligned.

By creating an equation for the alignment at $x$ minutes after 2 o'clock, $x$ minutes after 3 o'clock, and so on, we can solve the original problem.
Approach using a graph

The hour hand turns through 360° in 12 hours. The minute hand turns through 360° in an hour, and from 12 o'clock to the next 12 o'clock goes through 12 whole turns.

The following graph represents the movement of the two hands.

Problem 2: In the above diagram, add lines representing the movement of the minute hand, to complete the graph.

In the above diagram, if we take the horizontal axis as the \( x \)-axis and the vertical axis as the \( y \)-axis, we can write equations for the straight lines, using \( x \) and \( y \). Solve the original problems by finding the \( x \)-coordinates of the graph intersections.

Problem 3: Find the \( x \)-coordinate of intersection A in the diagram above, and find how many minutes after 1 o'clock the hands are aligned.
The following method is another approach to the problem, using a graph.

Measuring the angle from the 12 o'clock position on the clock face, from 12 o'clock to 1 o'clock, the minute hand turns from $0^\circ$ to $360^\circ$ in one hour. This movement is the same in each hour. Meanwhile, the hour hand starts from $0^\circ$ at 12 o'clock, and $30^\circ$ at 1 o'clock, turning through $30^\circ$ in each hour.

Let's plot these movements on a graph, and consider the original problem.

![Graph](image)

**Problem 4** In the above diagram, add lines representing the movement of the minute hand, to complete the graph.

In the above diagram, if we take the horizontal axis as the $x$-axis and the vertical axis as the $y$-axis and then by finding the $x$-coordinates of the intersections, we can solve the original problem.

**Problem 5** Find the $x$-coordinate of intersection B in the diagram above, and find how many minutes after 2 o'clock the hands are aligned.
Let's look at the original problem from the viewpoint of proportionality.

Suppose the hands are aligned at $x$ hours and $y$ minutes after 12 o'clock.
Since they are aligned at 12 o'clock, this is when $x = 0$ and $y = 0$.
When $x = 1$, the value of $y$ is the time it takes for the minute hand to catch up
with the hour hand that is 30° ahead. When $x = 2$, the value of $y$ is the time it
takes for the minute hand to catch up with the hour hand that is 60° ahead.
Looking at the problem this way, we see that as the value of $x$ is multiplied
by 2, 3, and so on, the value of $y$ is also multiplied by 2, 3, and so on, so
that $y$ is proportional to $x$.
Since the hands are aligned again 12 hours later at 12 o'clock, by regarding
12:00 as equivalent to 11:60, we see that when $x = 11$, $y = 60$.

**Problem 6** Draw a graph in the following diagram, showing the relation between $x$ and
$y$ above. Find an equation for $x$ and $y$.

A graph passes through the starting point
and (11, 60).

You will need to decide
the timing for standing.

**Problem 7** Using the approach above, what do we need to do to solve the original
problem? Explain the method.
How Many Turns on the Toilet Roll?

We use toilet paper by pulling it from a holder. How many times does the cardboard core in the roll turn until the roll is used up?

Consider toilet paper where the length of the roll is 65 m, and the other dimensions are as shown in the diagram on the right.

As the paper unwinds, one layer is removed for each turn of the roll. So we can tell how many times the core has rotated from the number of layers of paper.

To find how many layers, let's cut open the roll as shown below, and lay it flat.

When the paper is laid flat we expect to see a quadrangular prism whose base is a trapezoid. The part with the layers of paper is the base of the quadrangular prism.
1. The diagram below shows the base of the solid formed by cutting open the roll. Let the width a distance \( x \) cm from the upper side AD be \( y \) cm. Write an expression for \( y \) in terms of \( x \).

![Diagram of a roll of paper showing the base and dimensions](image)

From step 1 we see that if AB is a straight line, DC is also a straight line, and therefore the shape of the base is a trapezoid.

2. Since the total length is 65 m, and the shape of the base is a trapezoid, let's calculate the number of layers of paper forming the base, and thus find the number of turns.

3. By comparing the volume in the rolled up state with the volume when the paper is laid out flat, we can find the thickness of the paper, and based on the result we can determine the number of turns.

![Roll of paper with dimensions](image)

4. If we consider the case in which the core has zero diameter, when we cut open the paper as shown on the previous page, what is the shape of the base formed by the layers of paper?

5. Based on the argument above, explain the formula for finding the area of a circle.

---

Private Study 187
Traditional Patterns with Tiled Figures

The diagram on the right shows a traditional Japanese pattern known as "hemp leaves". We can view this pattern as consisting of a large number of congruent geometrical shapes laid together like tiles to fill the plane. Let's look at several examples.

1. Try drawing the "hemp leaves" pattern.

2. Look at the following traditional patterns, and say what shapes are tiled to make them.

- Basket weave
- Shippo ("Seven treasures")
- Hakone marquetry work (Kanagawa Prefecture)
Let's try tiling congruent figures.

We can fill the plane with congruent triangles thus.

(a) \hspace{1cm} (b)

We can make variants on the basic patterns (a) and (b) above to form new tilings.

3. If we view (a) as based on parallelograms, which are the basic parallelograms?

Look for examples of tilings around you, and analyze them in the same way as above.
Reading Braille

The photograph on the right shows the cover of the Ai-no-kobato Braille Calendar produced by the Ai-no-kobato Foundation. This has the message "Wishing you a heartfelt happy new year" written in Braille as follows.

A TA RA SHI I TO SHI NO A NA TA NI

TO SHI NO

A NA TA NI

KO KO RO WO KO ME TE O KU RI MA SU

In the above sample, the filled black circles represent raised dots.

Looking at how Braille works

Braille uses a grid of six dot positions, which are numbered as shown in the diagram on the right. For example, Japanese syllabic 'a' consists of dot number 1 only.

Based on the above message, let's try to fill in the Braille characters for the 50 syllables.

1. Enter the characters that appear in the message above in the table on the opposite page.

2. From the characters entered in step 1, try to predict the rest of the table (except for the ha-row).

3. The ha-row is constructed by the same rules you found in step 2. How do you think the ha-row is constructed?
### Syllabic table

<table>
<thead>
<tr>
<th>A</th>
<th>I</th>
<th>U</th>
<th>E</th>
<th>O</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>SA</th>
<th>SH</th>
<th>SU</th>
<th>SE</th>
<th>SO</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>NA</th>
<th>NI</th>
<th>NU</th>
<th>NE</th>
<th>NO</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>MA</th>
<th>MI</th>
<th>MU</th>
<th>ME</th>
<th>MO</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>RA</th>
<th>RI</th>
<th>RU</th>
<th>RE</th>
<th>RO</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>KA</th>
<th>KI</th>
<th>KU</th>
<th>KE</th>
<th>KO</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
<td>⠖</td>
<td>⠓</td>
<td>⠔</td>
<td>⠕</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>WA</th>
</tr>
</thead>
<tbody>
<tr>
<td>⠑</td>
</tr>
</tbody>
</table>

### Questions

4. What have you discovered about the rules for how the Braille syllabic table is made?

5. The Braille numerals are shown on the right. What do you notice about them?

6. Look for examples of Braille around you, and try to read it.

Where do you find Braille used?
Let's Make a Beam Balance

A beam balance is one type of scale for weighing. You can make one without special materials as follows.

Using the beam balance shown above, carry out the following experiment.

1. Clip an envelope in the clothes peg, and move the weight until the beam is horizontal and balances, then mark the position of the weight.
2. Put a sheet of notepaper in the envelope, then add more sheets one by one, each time marking the position of the weight.

From the results of the experiment, consider the following.

1. Measure the distances from the fulcrum to the marks, and investigate the relation to the number of sheets of notepaper.
2. Using the relation you have derived, try making a beam balance which you can actually use to weight objects.

In the scale museum in Matsumoto many beam balances are on display.
Experimenting with Dice for Random Numbers

The photograph on the right shows dice called "random number dice." These dice have two faces each numbered 0 to 9.

1. Using random number dice, try the following experiment.
   1. On graph paper with 1-mm divisions, draw coordinate axes as shown on the right. Then draw a fan shape with center at the origin and radius 100 mm, with a central angle of 90°.
   2. Throw the large and small dice. Using the smaller die for the tens digit and the larger die for the units digit, make a two-digit number. Do this twice.
   3. Taking the first number as the $x$-coordinate and the second number as the $y$-coordinate, plot the point with these coordinates on the graph paper with the fan shape. Take the value of the coordinates to be in millimeters.

2. Repeat steps 2 and 3 above and find the value:

$$\frac{\text{Number of points inside the fan}}{\text{Total number of points plotted}} \times 4$$
1. Calculate the following.
   ① 3x - 4y - 2x - 3y
   ② -5a² - a - a² + 6a
   ③ (2a + 9b) + (2a - 4b)
   ④ (7x² - 6x) - (-4x² - 5x)

2. Calculate the following.
   ① 4(2x + 3y)
   ② (-16a + 12b) × (-3)
   ③ (9a - 15b) ÷ (-3)
   ④ (25a² + 10a - 40) ÷ 5
   ⑤ 2(2x + y) + 3(x - 4y)
   ⑥ 5(a-b) + 2(-3a + 4b)
   ⑦ 4(3x² + 2x) - 2(5x² - x)
   ⑧ \[
   \frac{x + 3y}{4} + \frac{3x - y}{2}
   \]

3. Calculate the following.
   ① 7a × (-2b)
   ② 2xy × (3y)²
   ③ (-24xy) ÷ (-6x)
   ④ 15a² ÷ (-3a)
   ⑤ \[
   \frac{9}{14} a^2 b \div \frac{6}{7} ab
   \]
   ⑥ 18x³ ÷ (-9x²) × 2x

---

Let's try!

Albert has lots of 50-yen and 80-yen postage stamps at home. He is investigating which face value amounts he can make with different combinations. Are there any amounts over 200 yen that it is not possible to make? Find these amounts. You only need to consider multiples of 10 yen.
Practicing Solving Simultaneous Equations

1. Solve the following sets of simultaneous equations.
   
   (1) \[
   \begin{align*}
   2x + y &= 7 \\
   2x - y &= 1 \\
   \end{align*}
   \]

   (2) \[
   \begin{align*}
   x + 4y &= -1 \\
   3x + 4y &= 5 \\
   \end{align*}
   \]

   (3) \[
   \begin{align*}
   -x + 5y &= 5 \\
   x - 3y &= 1 \\
   \end{align*}
   \]

   (4) \[
   \begin{align*}
   4x - 3y &= 19 \\
   3x + y &= 11 \\
   \end{align*}
   \]

   (5) \[
   \begin{align*}
   3x + 2y &= 7 \\
   5x + 4y &= 13 \\
   \end{align*}
   \]

   (6) \[
   \begin{align*}
   2x + 5y &= -16 \\
   -3x + 4y &= 24 \\
   \end{align*}
   \]

2. Solve the following sets of simultaneous equations.
   
   (1) \[
   \begin{align*}
   4x + y &= -5 \\
   y &= -2x + 1 \\
   \end{align*}
   \]

   (2) \[
   \begin{align*}
   x &= 3y - 4 \\
   2x - 3y &= 1 \\
   \end{align*}
   \]

   (3) \[
   \begin{align*}
   y &= 2x - 9 \\
   y &= -2x + 7 \\
   \end{align*}
   \]

   (4) \[
   \begin{align*}
   2x + 7y &= 2 \\
   2x &= 3y + 22 \\
   \end{align*}
   \]

3. Solve the following sets of simultaneous equations.
   
   (1) \[
   \begin{align*}
   x - 5y &= -1 \\
   3(x + 4) - 2y &= -4 \\
   \end{align*}
   \]

   (2) \[
   \begin{align*}
   7x - 3(x + y) &= 6 \\
   5x + 2y &= 19 \\
   \end{align*}
   \]

Let's try!

The pheasants and rabbits problem described in "Window on math" on page 40 was solved by Sun Zi using the following method.

(Number of rabbits) = (Number of legs) ÷ 2 - (Number of heads)

(Number of pheasants) = (Number of heads) - (Number of rabbits)

Explain why this method works.
1. Patrick left his house at 10 o'clock, stopping at a bookstore on the way, and then going to the library. The diagram on the right is a graph showing the relation between the time elapsed since Patrick left home and the distance he has walked.

   (1) At what speed, in meters per minute, did Patrick walk from home to the bookstore?
   (2) His sister left home at 10:30, and set off for the library by bicycle. If her speed is 12 km/hour, find the time at which she catches up her brother.

2. A water tank has two supply pipes A and B, and one drain pipe. The supply pipes A and B are turned on at the same time, and start filling the tank. Four minutes after starting to fill the tank, it is realized that the drain pipe is open. The drain pipe is then closed. Four minutes later supply pipe A is turned off. The diagram on the right shows a graph of the relation between the elapsed time after starting to fill the tank and the amount of water in the tank.

   (1) At what rate was water supplied from supply pipe A, in liters per minute?
   (2) How fast does water drain from the drain pipe, in liters per minute?

---

Express trains travel between stations A and B in either direction in 2 hours 30 minutes. The first express leaves each station at 6:00 AM, and thereafter trains leave at one-hour intervals. As the train leaving station A at 8:00 AM makes its journey, how many times does it meet a train coming in the opposite direction before reaching station B?
Practicing Finding Angles

1 Answer the following questions.
   ① Find the sum of the interior and exterior angles of a nonagon (polygon with nine sides).
   ② How many degrees is an interior angle of a regular decagon (ten sides)? How many degrees is an exterior angle?
   ③ How many sides does a polygon have if the sum of the interior angles is 1800°

2 In the following diagrams, find the size of \( \angle x \).
   ① \( \ell \parallel m \) \hspace{1cm} ② \( \ell \parallel m \)
   \[ \ell \hspace{0.5cm} \begin{array}{c} 55° \end{array} \hspace{1cm} \begin{array}{c} \angle x \end{array} \hspace{1cm} \begin{array}{c} \angle x \end{array} \hspace{1cm} \begin{array}{c} 140° \end{array} \]

3 In the following diagrams, find the size of \( \angle x \).
   ① \hspace{1cm} ② \hspace{1cm} ③ \hspace{1cm} ④ \hspace{1cm} ⑤ \hspace{1cm} ⑥
   \[ \begin{array}{c} 50° \end{array} \hspace{1cm} \begin{array}{c} \angle x \end{array} \hspace{1cm} \begin{array}{c} 36° \end{array} \hspace{1cm} \begin{array}{c} 50° \end{array} \hspace{1cm} \begin{array}{c} 130° \end{array} \]

Let's try!

Using a square sheet of origami paper, try to fold an angle of 15°. Keep the number of folds as low as possible!
Proving Geometrical Properties

1. Quadrilateral ABCD in the diagram on the right is a trapezoid, with AD \parallel BC, and \angle ADC = \angle ACD. Point E lies on the diagonal AC such that \angle BAC = \angle ADE.
   Prove that BC = AE.

2. In \square ABCD as shown in the diagram on the right, the intersection of the bisector of \angle BAD and the side BC is point E.
   Prove that EC + CD = AD.

3. In the diagram on the right, E and F are points on the diagonal BD of \square ABCD such that \angle BAE = \angle DCF. Prove that AECF is a parallelogram.

Let's try!

The diagram on the right includes three congruent squares joined together. Can you find the sum of the angles \angle x, \angle y, and \angle z drawn in the diagram?

Well, \angle x is 45', but...
Finding Probabilities

1. I throw two dice of different sizes, and take the number on the larger die as \( a \), and the number on the smaller die as \( b \). If point A has coordinates \((2, a)\) and point B has coordinates \((4, b)\), find the following probabilities.
   - Probability that the straight line \( AB \) is parallel to the \( x \)-axis
   - Probability that the straight line \( AB \) passes through the origin
   - Probability that the slope of the straight line \( AB \) is -1

2. As shown in the diagram on the right, ABC is an equilateral triangle of side 1 cm. Point P is at the vertex A. For each time that a coin is tossed, point P moves along the sides of the triangle through vertices A, B, C in sequence. It moves 2 cm if the coin toss is head and 1 cm if coin toss is tail.
   - The coin is tossed twice, and both times are heads. Now which vertex has the point P reached?
   - If the coin is tossed three times, find the probability that the final position of point P is the vertex B.

Let's try!

In a set of yes/no questions, we found the following tendency in the answers to the questions.

1. "Yes" answer is more common.
2. A "Yes" or a "No" answer never appears three or more times consecutively.

There are five questions, and I do not know the answer to any of them. What sequence of yes/no answers should I give to increase the likely proportion of correct answers?
Answers to chapter summary problems

Chapter 1 — Calculating with Formulas

Chapter summary problems A — p.22

1
1. \(-2a + 2b\)
2. \(3x - y\)
3. \(7x - 12y\)
4. \(4x^2\)

2
1. \(6a - 9b\)
2. \(-2a - 8b\)
3. \(a - 3b\)
4. \(-2a^2 + 3x\)
5. \(10a - 5b\)
6. \(2x^2 + 3\)

3
1. \(-18x^2\)
2. \(4x^2\)
3. \(-\frac{a}{2}\)
4. \(-9x\)
5. \(15xy\)
6. \(2a\)

4
1. \(-8\)
2. \(4\)

5
1. \(y = \frac{3}{4}x + \frac{1}{2}\)
2. \(a = 2m - b\)

6

We can write the sequence of 3 consecutive even numbers as \(2n - 2, 2n, 2n + 2\) where \(n\) is the middle integer. Therefore, the sum of these is \((2n - 2) + 2n + (2n + 2) = 6n\) which explains why it is a multiple of 6.

Chapter summary problems B — p.23

1
1. \(\frac{8x + y}{6}\)
2. \(2x^2 - 2x - 4\)

2
1. \(-2x + 13y\)
2. \(x + 6y\)

3
19.8 cm²

4
We can explain this by letting the central number be \(x\); then the sum of the three numbers is \((x - 7) + x + (x + 7) = 3x\)

5
The answer is double the number used in step 3.

Chapter 2 — Simultaneous Equations

Chapter summary problems A — p.41

1
1. \(x = 3, y = -2\)
2. \(x = 1, y = 5\)
3. \(x = -\frac{3}{2}, y = 0\)
4. \(x = 4, y = 9\)
5. \(x = 2, y = 6\)
6. \(x = 6, y = 5\)

2
1. \(a = 3, b = 5\)
2. Eight 50-yen stamps
   Seven 80-yen stamps

4
31

5
20 boys and 16 girls

Chapter summary problems B — p.42

1
1. \(x = -10, y = -8\)
2. \(x = 1, y = -9\)

2
18 girls and 16 boys

3
1. A: 12 km/hour
   B: 4 km/hour

Chapter 3 — Linear Functions

Chapter summary problems A — p.76

1
1. \(x = 3, y = 14\)
   \(x = 2, y = -1\)
2. \(-15\)
3. \(-4 \leq y \leq 17\)

2
1. \(y = \frac{2}{5}x + 1\)
2. \(y = 2x - 1\)
3. \(y = -3x + 2\)
3 (a) \( y = 3x \)
(b) \( y = -\frac{2}{3}x - 1 \)
(c) \( y = \frac{1}{2}x + 1 \)
(d) \( y = -x + 7 \)
(2) \( x = 4, \ y = 3 \)
(3) \( \left(-\frac{3}{11}, -\frac{9}{11}\right) \)

4. 3930 yen

Chapter 4 - Parallel Lines and Congruent Figures

Chapter summary problems B - p. 107

1. \( \angle a, \angle e, \angle d, \angle f \cdots 140^\circ \)
2. \( \angle e, \angle f, \angle h \cdots 40^\circ \)
3. (1) Regular octagon
   (2) 3240°
   (3) Nonagon
4. (1) Hypothesis \( PA = PB \)
   \( AM = BM \)
   Conclusion \( \angle AMP = \angle BMP \)
   (2) \( \triangle PAM \) and \( \triangle PBM \)
   (3) 3 sides are all equal.

Chapter summary problems B - p. 108

1. \( \angle a, \angle e, \angle d, \angle f \cdots 140^\circ \)
2. \( \angle e, \angle f, \angle h \cdots 40^\circ \)
3. (1) \( 10^\circ \)
   (2) \( 20^\circ \)
4. Since 2 sides and the included angle are equal, \( \triangle PAM = \triangle PBM \)
   Therefore, \( \angle AMP = \angle BMP \)

Chapter 5 - Properties of Geometrical Figures

Chapter summary problems A - p. 145

1 In triangles \( \triangle ABC \) and \( \triangle ACD \)
   \( AB = AC \)
   \( \angle BAC = \angle CAD \)
   \( AP \) is a shared side
   Therefore, \( \triangle ABP = \triangle ACP \)
   Hence \( PB = PC \)

2 (1) In triangles \( \triangle DBC \) and \( \triangle EBC \)
   \( AB = AC \)
   \( DB = \frac{1}{2} AB, \ EC = \frac{1}{2} AC \)
   Therefore \( DB = EC \)
   Since the base angles of an isosceles triangle are equal,
   \( \angle DBC = \angle ECB \) \( \cdots (1) \)
   \( BC \) is a shared side \( \cdots \) \( (2) \)
   From (1), (2), and (3), since two sides and the included angle are equal,
   \( \triangle DBC = \triangle ECB \)
   Therefore \( BE = CD \)
   (2) From (1), \( \angle FBC = \angle FCB \)
   Since the two angles are equal, \( \angle FBC \)
   is an isosceles triangle.

3 In \( \angle ABCD \), diagonal \( AC \) passes through the midpoint of \( BD \).
   In \( \angle A'B'C'D \), diagonal \( A'C' \) passes through the midpoint of \( BD \).
   Therefore, \( AC, A'C' \) and \( BD \) all intersect at the midpoint of \( BD \).

Answer 201
In triangles \( \triangle ABE \) and \( \triangle CDF \), since opposite sides of a parallelogram are equal,
\[ AB = CD \quad \cdots \quad (1) \]
And since alternate angles of parallel lines are equal,
\[ \angle ABE = \angle CDF \quad \cdots \quad (2) \]
By hypothesis,
\[ \angle AEB = \angle CFD = 90^\circ \quad \cdots \quad (3) \]
Since these are right-angled triangles, with the hypotenuse and one acute angle equal,
\[ \triangle ABE \cong \triangle CDF \]
Therefore, \[ AE = CF \quad \cdots \quad (4) \]
And since \( \angle AEF = \angle CFE = 90^\circ \), and these are equal alternate angles, \( \triangle AFE \cong \triangle CFE \quad \cdots \quad (5) \)
Since one pair of opposite sides are parallel and the same length, \( \triangle ABCF \) is a parallelogram.

5. \( \begin{align*}
1 & \quad 90^\circ \\
2 & \quad 30^\circ \\
3 & \quad 100^\circ 
\end{align*} \)

Chapter summary problems B — p. 146

1. \( \begin{align*}
& \quad 1 \quad 2 \quad 3 \\
& \quad 1 \quad 2 \\
& \quad 3
\end{align*} \)

By hypothesis,
\[ EF = DE, \ AE = FC \]
Since the diagonals intersect at their midpoints, \( \triangle ACF \) is a parallelogram.

2. \( \begin{align*}
& \quad 2 \\
& \quad 1 \\
& \quad 1
\end{align*} \)

From \( \triangle ABCF \) are parallel and the same length, \( \triangle BDF \) is a parallelogram.

Therefore, \[ DF \parallel BC, \ DE = \frac{1}{2} BC \]

Chapter 6 — Probability

Chapter summary problems A — p. 166

1. \( \begin{align*}
& \quad 1 \quad 2 \quad 3 \\
& \quad 1 \\
& \quad 3 \\
& \quad 4
\end{align*} \)

Examples: "odd", "even", "a multiple of four", and so on

Chapter summary problems B — p. 167

1. \( \begin{align*}
& \quad 1 \\
& \quad 2 \\
& \quad 3
\end{align*} \)

Drawing a \( 2 \) and a \( 3 \) is more likely.

3. \( \frac{3}{5} \)
Answers to Development Questions

### Multiplying and dividing powers of \( a \)

**Problem 1**
\[
(a^2)^3 = a^2 \times a^2 \times a^2 = a^{2+2+2} = a^6
\]

**Problem 2**
Since \( a^1 : a^1 = 1 \), we can see that \( a^0 = 1 \).

### Simultaneous Equations of the Form \( A = B = C \)

**Problem 1**
Adults: 100 yen, child: 50 yen

**Problem 2**
1. \( x = 2, y = 1 \)
2. \( x = 2, y = 2 \)

### Graphs in Step Form

**Problem 1**
Example: Rail fares and similar

**Problem 1**
Graph of \( x = h \)

### Conditions for a Rectangle, Rhombus, and Square

1. (a), (c)
2. (b), (d)
3. (b), (d)
4. (a), (c)

---

Geometrical Figures Maintaining a Constant Angle

**The path is part of a circle.**

**Problem 1**

**Problem 2**

**Problem 3**
1. \( \angle APB = \angle a \)
2. \( \angle APB > \angle a \)
3. \( \angle APB < \angle a \)

In all cases, \( \angle APB \) is equal to \( \angle C \).

**Problem 4**
If approaches \( \angle ABT \).

**Problem 5**
The exterior angle of quadrilateral \( APBC \) at vertex \( P \).

**Problem 6**
1. Draw diagonals \( AB \) and \( CP \) of quadrilateral \( APBC \).

By the inscribed angle theorem,

\[
\angle ACP = \angle ABP
\]

\[
\angle PCB = \angle PAB
\]

And from the relation between interior and exterior angles of a triangle,

\[
\angle APE = \angle ABP + \angle PAB
\]

\[
= \angle ACP + \angle PCB
\]

\[
= \angle ACB
\]

\[
(2) \quad \angle APE + \angle APB = 180^\circ
\]

From (1), since \( \angle ACB = \angle APE \),

\[
\angle ACB = \angle APB = 180^\circ
\]

**Problem 7**
Extend \( BO \), and let \( C' \) be the intersection with the circle. Since \( OB \perp BT \),

\[
\angle ABT = 90^\circ = \angle C'BA
\]

But \( \triangle CAB \) is a right-angled triangle, and therefore

\[
\angle AC'B = 90^\circ = \angle C'BA
\]

Therefore

\[
\angle ABT = \angle AC'B
\]

From the inscribed angle theorem,

\[
\angle AC'B = \angle ACB
\]

\[
\angle ABT = \angle ACB
\]

Answer 203
Probability of an Event Not Occurring  p.168

Equations with Three Unknowns  p.170~173

Problem 1
The sum of \( x \) and \( y \) is 2 more than the sum of \( x \) and \( z \). This means that \( y \) is 2 more than \( x \).

Problem 2
\[ x = 4, \quad y = -1, \quad z = -3 \]

Problem 3
(1) Subtract (3) from (1).
(2) For example, substitute the value of \( y \) into (1) to find the value of \( x \).

Finding the Number of Cases  p.174~177

Problem 1
Since each team plays every team but itself, we can find the answer from
\[(\text{number of teams} - 1) \times (\text{number of teams})\]

Problem 2
Entries \( a \) and \( b \) in the table are now seen as the same, so we can divide the total entries in the table by 2.
\[ \frac{12}{2} = 6 \quad \text{Answer 6 matches} \]

Problem 3
\[ 7 \times 6 \div 2 = 21 \quad \text{Answer 21 dots} \]

Problem 4
\[ 5 \times 4 = 20 \quad \text{Answer 20 numbers} \]

Making Predictions from Data  p.178~181

Problem 1

Problem 2

Problem 3
Omitted (The actual full bloom date was April 26.)

Problem 4
(1) Omitted
(2) If the total sunlight in July of the previous year is higher, there is a tendency for a higher pollen count.
Answers to Supplementary Problems

Practicing Algebraic Calculations — p. 194

1. 1. \( x - 7y \)  2. \(-6a^2 + 5a\)
   3. \(4a + 5b\)  4. \(11x^2 - x\)

2. 1. \(8x + 12y\)  2. \(12a - 9b\)
   3. \(3a + 5b\)  4. \(5a^2 + 2a - 8\)
   5. \(7x - 10y\)  6. \(-a + 3b\)
   7. \(2x^2 + 10x\)  8. \(\frac{7x + y}{4}\)

3. 1. \(16ab\)  2. \(18xy^2\)
   3. \(4y\)  4. \(-5a\)
   5. \(3a\)  6. \(4x^2\)

Practicing Solving Simultaneous Equations — p. 195

1. 1. \(x = 2, y = 3\)
   2. \(x = 3, y = 1\)
   3. \(x = 10, y = 3\)
   4. \(x = 4, y = 1\)
   5. \(x = 1, y = 2\)
   6. \(x = 8, y = 0\)

2. 1. \(x = 3, y = 7\)
   2. \(x = 5, y = 3\)
   3. \(x = 4, y = 1\)
   4. \(x = 8, y = -2\)

3. 1. \(x = 2, y = 1\)
   2. \(x = 3, y = 2\)

Solving Problems Using Graphs — p. 196

1. 1. 50 m/min
   2. 15 l/min

2. 1. 16 \(\frac{1}{2}\)  2. 12 \(\frac{1}{3}\)  3. 9 \(\frac{1}{3}\)

2. 1. Vertex 11 \(\frac{3}{8}\)

Practicing Finding Angles — p. 197

1. 1. Sum of interior angles: \(1260^\circ\)
   2. Sum of exterior angles: \(360^\circ\)
   3. 12 sides

2. 1. 125\(^\circ\)  2. 30\(^\circ\)  3. 50\(^\circ\)

3. 1. 80\(^\circ\)  2. 57\(^\circ\)  3. 78\(^\circ\)
   4. 72\(^\circ\)  5. 25\(^\circ\)  6. 100\(^\circ\)

Proving Geometrical Properties — p. 198

1. In \(\triangle ABC\) and \(\triangle DEA\)
   \[
   \begin{align*}
   AC &= DA \\
   \angle BCA &= \angle EAD \\
   \angle BAC &= \angle EDA
   \end{align*}
   \]

   Hence, \(\triangle ABC = \triangle DEA\)

2. \(\triangle ABE\) is an isosceles triangle
   Therefore, \(BA = BE = CD\)
   \[
   \begin{align*}
   AD &= BC \\
   \angle BAE &= \angle DAC \\
   \angle EAC &= \angle FCD
   \end{align*}
   \]

3. Since \(\triangle ABE = \triangle CDF\)
   \[
   \begin{align*}
   AE &= CF \quad \text{[1]} \\
   \angle AEF &= \angle CDF \\
   \angle AEF &= \angle CDF
   \end{align*}
   \]
   From (1) and (2), we can say that \(ABCF\) is a parallelogram.

Finding Probabilities — p. 199

1. 1. \(\frac{1}{6}\)  2. \(\frac{1}{12}\)  3. \(\frac{1}{9}\)

2. 1. Vertex 11 \(\frac{3}{8}\)

Answer 205
インデックス

[a-o]
1次関数 (Linear function) 50
1次式 (Linear expression) 7
円周角 (Inscribed angle) 142

[ka-ko]
解 (Solution) 28
外角 (Exterior angle) 82
確率 (Probability) 155

[88-90]

逆 (Converse) 119

結論 (Conclusion) 101

項 (Term) 6

3x^2 + 2xy + (-5)

合同 (≡) (Congruence (≡)) 94, 95

.

両角 (Alternate angles) 87

次数 (Degree) 7

\( x^2 + 2xy + 4 \)

2 3  \( \rightarrow \) of an expression

斜辺 (Hypotenuse) 120


関数 (Function) 48
消去 (Elimination) 28
証明 (Proof) 90
切片 (Intercept) 56
$
y= \text{ax+b}
\$
定義 (Definition) 112
底辺 (Base) 112
定理 (Theorem) 114
同位角 (Corresponding angles) 87
同様に確からしい (Equally likely) 157
同類項 (Similar terms) 8
対角 (Diagonal) 124
対頂角 (Vertically opposite angles) 86
入法 (Substitution) 32
対辺 (Opposite sides) 124
対頂角 (Vertically opposite angles) 86
多項式 (Polynomial) 6
$2x+5, 3a^2+4ab+1$
内角 (Interior angles) 82
単項式 (Monomial) 6
$2x, \frac{1}{3}a^2$
方程式のグラフ (Graph of an equation) 66
頂角 (Apex) 112
方程式のグラフ (Simultaneous equations) 26
底辺 (Base angle) 112
変化の割合 (Rate of change) 52
四角形ABCD 124
$2x+y=24$
$2x^2+3x^2-3$

207
About Wasan

Japanese mathematicians made their own independent advances during the Edo period. Their work is called "wasan". Let's look at the work of Seki Kowa, a distinguished practitioner of "wasan".

Seki Kowa

Seki Kowa (c. 1642 - 1708), also known as Seki Takakazu, laid the foundation for the development of mathematics in the Edo period. He is hailed as the "Saint" of "wasan". Among his achievements is his work on $\pi$.

Katsuyosanpo (Wasan Research Institute, Kyoto) contains the works of Seki Kowa collected by his pupil. The photograph below shows part of the calculation of $\pi$. 

Seki Kowa
(Kouju Collection, Toyama Prefecture)
Fill in the boxes with words or symbols to complete the following statements.

**Parallel lines and angles**
- Vertical angles are \( \square \).
- Relation between parallel lines and angles:
  - If two straight lines are parallel, then corresponding angles and alternate angles are \( \square \).
  - If corresponding angles or alternate angles are equal, the two straight lines are \( \square \).

**Congruence**
- Properties of vertical angles:
  - Vertical angles are \( \square \).
- Properties of congruent figures:
  - Corresponding \( \square \) and angles are equal.

**Isosceles triangles**
- Properties of isosceles triangles:
  - The base angles are \( \square \).
  - The bisector of the apex is the \( \square \) bisector of the base.
- Conditions for an isosceles triangle:
  - Two angles of the triangle are \( \square \).

**Circles**
- Inscribed angle theorem:
  - The size of an inscribed angle intercepting a particular arc of the circle is \( \square \) and is \( \square \) the central angle for that arc.
Conditions for congruent triangles:
- Sides are equal.
- Two sides and the included angle are equal.
- Two sides and the angles at each end are equal.

Conditions for congruent right-angled triangles:
- The hypotenuse and one side are equal.
- The hypotenuse and another side are equal.

Properties of a parallelogram:
- Both pairs of opposite sides are equal.
- Both pairs of opposite angles are equal.
- The diagonals intersect at their midpoints.
- One pair of opposite lengths are parallel and their lengths are equal.
本書は、東京書籍株式会社発行の検定済教科書『新編 新しい数学 2』（代表著作者：杉山、吉茂・侯野・博）平成17年2月3日検定済を、同社の承諾を得て、学校法人太田国際学園が英訳し、発行したものです。

本書の内容についてのすべての責任は、学校法人太田国際学園にあります。