Chapter 1 Square Roots

1

Square Roots

Let's draw squares of different areas, and find the length of the side in each case.

1. On the page on the right, draw squares having the following areas.
   * a) 1 cm²  b) 4 cm²  c) 2 cm²  d) 5 cm²

2. Try drawing squares of other areas.

3. For each square you draw in 1 and 2, measure the length of the side, and complete the following table.

<table>
<thead>
<tr>
<th>Square</th>
<th>a</th>
<th>b</th>
<th>c</th>
<th>d</th>
</tr>
</thead>
<tbody>
<tr>
<td>Area (cm²)</td>
<td>1</td>
<td>4</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>Length of a side (cm)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. Calculate $1.4^2$ and $1.5^2$, and check that the side of a square of area 2 cm² must be between 1.4 cm and 1.5 cm.

If the side of a square of area 2 cm² is $x$ cm, then we have $x^2 = 2$. In our investigation we find that $x = 1.4142135\ldots$

Check that the value of $x$ is between 1.4142135 and 1.4142136.
1. Square Roots

Generally, if squaring a number \( x \) results in \( a \), or if \( x^2 = a \), we say that \( x \) is the square root of \( a \).

For example,
\[
3^2 = 9, \quad (-3)^2 = 9
\]
Therefore both 3 and -3 are square roots of 9.

Example 1

Since \( \left( \frac{4}{9} \right)^2 = \frac{16}{81}, \quad \left( -\frac{4}{9} \right)^2 = \frac{16}{81} \)

the square roots of \( \frac{16}{81} \) are \( \frac{4}{9} \) and \( -\frac{4}{9} \).

Check: Give the square roots of the following numbers.
1. 16
2. 49
3. \( \frac{4}{25} \)
4. 0.01

The only number that results in zero when squared is zero itself. Also, no number squared can give a negative result. Therefore, a negative number does not have a square root.

- **Square roots**

| 1 | A positive number has two square roots, which have the same absolute value and opposite signs. |
| 2 | Zero has the single square root zero. |

For a positive number \( a \), we write the two square roots as follows:
\[
\sqrt{a} \quad \text{is the positive square root}
\]
\[
-\sqrt{a} \quad \text{is the negative square root}
\]

Also we have \( \sqrt{0} = 0 \).

The symbol \( \sqrt{\cdot} \) is called the square root sign, and we read, \( \sqrt{a} \) as "the root of \( a \)".

The symbol \( \sqrt{\cdot} \) is also known as a radical sign. The number under the radical sign is called radicand. Expressions consisting of radical sign and radicands are called radicals.
The square roots of 2 are $\sqrt{2}$ and $-\sqrt{2}$.

We can write $\pm \sqrt{2}$ to refer to the square roots of 2, $\sqrt{2}$ and $-\sqrt{2}$, together. We pronounce this notation as the "positive and negative root two".

Check 2 Write the square roots of the following numbers using the square root sign.

1. $3$
2. $10$
3. $0.5$
4. $\frac{3}{7}$

We can write the root of some numbers without using the square root sign.

Example 3

1. $\sqrt{25} = 5$
2. $\sqrt{(-5)^2} = \sqrt{25} = 5$

Check 3 Write the following numbers without using the square root sign.

1. $\sqrt{36}$
2. $-\sqrt{81}$
3. $\sqrt{(-7)^2}$

Problem 1 Write the following numbers without using the square root sign.

1. $\sqrt{\frac{4}{9}}$
2. $\sqrt{1}$
3. $-\sqrt{7^2}$

The following equations hold for any positive number $a$.

$$(\sqrt{a})^2 = a$$

$$( - \sqrt{a})^2 = a$$

Problem 2 Find the values of the following expressions.

1. $(\sqrt{7})^2$
2. $(-\sqrt{13})^2$
3. $(\sqrt{16})^2$
4. $(\sqrt{\frac{9}{5}})^2$
Comparing Square Roots

We can show the values of $\sqrt{2}$ and $\sqrt{5}$ on a number line, as shown in the diagram on the right. From this diagram we can see that $\sqrt{2} < \sqrt{5}$.

As the area of a square increases, the length of a side also increases. In general as the positive number increases, its square root also increases.

In other words, we have the following facts when comparing two square roots.

<table>
<thead>
<tr>
<th>Comparing square roots</th>
</tr>
</thead>
<tbody>
<tr>
<td>If $a$ and $b$ are positive numbers, and $a &lt; b$, then:</td>
</tr>
<tr>
<td>$\sqrt{a} &lt; \sqrt{b}$</td>
</tr>
</tbody>
</table>

Example 4 Comparing $\sqrt{4}$ and $\sqrt{21}$

Since $4^2 = 16$ and $(\sqrt{21})^2 = 21$, and $16 < 21$, then $\sqrt{16} < \sqrt{21}$.

That is, $4 < \sqrt{21}$

Check 4 Compare the following sets of numbers using inequality signs.

1. $\sqrt{13}$, $\sqrt{15}$
2. $5$, $\sqrt{23}$

Problem 3 Compare the following sets of numbers using inequality signs.

1. $\sqrt{0.1}$, $0.1$
2. $-\sqrt{12}$, $-\sqrt{17}$
3. $2$, $3$, $\sqrt{5}$
Let's look at the value of $\sqrt{5}$.

$2.2^2 = 4.84$, and $2.3^2 = 5.29$, from which we have that $2.2 < \sqrt{5} < 2.3$

Problem 4
Find $2.21^2$, $2.22^2$, ... and so on in sequence, to determine the second decimal place of $\sqrt{5}$.

You can also find an approximate value of $\sqrt{5}$ by using the key on your calculator.

Problem 5
Using the key on your calculator, find an approximate value of $\sqrt{5}$.

If we write $\sqrt{5}$ as a decimal:

$\sqrt{5} = 2.2360679774997896964091736\ldots$

And this continues without limit. We also know that it is not possible to express $\sqrt{5}$ as a fraction.

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Window on math — Square root values and ways to remember them

$\sqrt{2} = 1.41421356237\ldots$

$\sqrt{3} = 1.73205080756\ldots$

$\sqrt{4} = 2$

$\sqrt{5} = 2.23606797749\ldots$

$\sqrt{6} = 2.44948974278\ldots$

A number like $\sqrt{5}$ which cannot be expressed as a fraction, is called an irrational number.

Numbers which can be expressed as fractions are called rational numbers.
Prime Factorization

Let's look at numbers with a root sign, to see if they are integers.

For example \( \sqrt{196} \) is an integer. To confirm this, we can check whether 196 is the square of an integer.

If we write 196 as the product of smaller natural numbers we get the following:

\[
196 = 2 \times 98 = 2 \times 2 \times 7 \times 7 = (2 \times 7)^2 = 14^2
\]

Therefore, since 196 is the square of 14, \( \sqrt{196} = 14 \).

Check 1 Use the method above to determine whether \( \sqrt{225} \) is an integer.

We can write 196 as \( 2 \times 98 \), \( 2 \times 2 \times 7 \times 7 \), or as the product of other sets of natural numbers. We call these numbers the factors of 196.

Example 1 Since we can write \( 12 = 2 \times 6 \), 2 and 6 are factors of 12.

The numbers 2, 3, 5, 7, and so on, which cannot be written as products of smaller natural numbers are called prime numbers (or sometimes just "primes"). A prime number is one with no divisors other than 1 and itself. However, we do not consider 1 a prime number.

Check 2 Give all the prime numbers between 10 and 20.

A factor which is a prime number is called a prime factor. The process of breaking down a natural number into a product of primes is called prime factorization.

Example 2 The prime factorization of 30 is as follows.

\[
30 = 2 \times 3 \times 5
\]

\[
10 \quad - \quad \text{Square Roots} \quad \Rightarrow p.25 \quad \text{The sieve of Eratosthenes}
\]
Problem 1  Enter the appropriate number in the boxes on the diagram on the right.

Whatever order we carry out prime factorization, the result will be the same.

Example 3  Let's find the prime factors of 28.
1. Divide 28 by the prime numbers in sequence.
2. Form the product of the prime factors.

Check 3  Find the prime factors of the following numbers.
1. 24
2. 50
3. 66
4. 140

By taking the prime factors of a number we can see how it can be expressed as a product of primes. Using this fact, let's think about the following problem.

Problem 2  Find the square roots of 144, using prime factorization.

Example 4  We want to multiply 63 by the smallest natural number so that the product is a perfect square*. What number should we multiply 63 by? What is the product?

Answer  The prime factorization of 63 is:

\[ 63 = 3^2 \times 7 \]

And to make this into a perfect square, we multiply \(3^2 \times 7\) by 7 to give:

\[ 3^2 \times 7 \times 7 = (3 \times 7)^2 = 21^2 \]

Answer: Multiply by 7 to obtain the square of 21.

Problem 3  We want to multiply 108 by the smallest number so that the product is a perfect square. What number should we multiply 108 by? What is the product? The product should be a perfect square.

* Perfect square: a number which is the square of a natural number.
Basic Exercises

Square roots

1. Find the square roots of the following numbers.
   
   1. $\sqrt{225}$
   2. $\sqrt{100}$
   3. $\sqrt{400}$
   4. $\sqrt{\frac{3}{5}}$

Numbers with root signs

2. Write each of the following numbers without using root signs.

   1. $\sqrt{64}$
   2. $\sqrt{100}$
   3. $\sqrt{0.19}$
   4. $(\sqrt{6})^2$

Comparing square roots

3. For the following sets of numbers, write an inequality showing which of the two is larger.

   1. $\sqrt{61}$, $\sqrt{70}$
   2. $4$, $\sqrt{18}$

Prime factorization

4. Find the prime factors of the following numbers.

   1. 42
   2. 52
   3. 90

Window on math — Origin of the root sign

The word "root" comes from Latin, radix. The root sign, $\sqrt{}$, is thought to have originated as a stylized version of the initial 'r' in radix.

16th century mathematical text using root signs
Given a rectangle whose length is \( \sqrt{5} \) cm and whose width is \( \sqrt{2} \) cm. Can we calculate the area of the rectangle as \( \sqrt{2} \times \sqrt{5} \)?

Using a calculator, find the approximate values of \( \sqrt{2} \times \sqrt{5} \) and \( \sqrt{2} \times 5 \). Then compare them.

For any positive numbers \( a \) and \( b \), we have

\[
\sqrt{a} \times \sqrt{b} = \sqrt{ab}
\]

Let's check this when

\( a = 2, \ b = 5 \)

\[
(\sqrt{2} \times \sqrt{5})^2 = (\sqrt{2} \times 5) \times (\sqrt{2} \times \sqrt{5})
\]
\[
= (\sqrt{2})^2 \times (\sqrt{5})^2
\]
\[
= 2 \times 5
\]

So \( \sqrt{2} \times \sqrt{5} \) is the square root of \( 2 \times 5 \).

Also, since \( \sqrt{2} \) and \( \sqrt{5} \) are both positive, \( \sqrt{2} \times \sqrt{5} \) is also positive.

Therefore, since \( \sqrt{2} \times \sqrt{5} \) is the positive square root of \( 2 \times 5 \), we have

\[
\sqrt{2} \times \sqrt{5} = \sqrt{2 \times 5}
\]

The same relation holds for the quotient

\[
\frac{\sqrt{2}}{\sqrt{5}} = \sqrt{\frac{2}{5}}
\]
So we have the following facts about the multiplication and division of square roots.

| Products and quotients of square roots |

<table>
<thead>
<tr>
<th>If ( a ) and ( b ) are positive numbers:</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \sqrt{a} \times \sqrt{b} = \sqrt{ab} )</td>
</tr>
<tr>
<td>2. ( \frac{\sqrt{a}}{\sqrt{b}} = \sqrt{\frac{a}{b}} )</td>
</tr>
</tbody>
</table>

We can write \( \sqrt{a} \times \sqrt{b} \) as \( \sqrt{ab} \), omitting the multiplication sign. Similarly we write \( a \times \sqrt{b} \) as \( a\sqrt{b} \).

**Let's evaluate expressions including root signs.**

<table>
<thead>
<tr>
<th>Example 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( \sqrt{3} \times \sqrt{2} = \sqrt{3 \times 2} = \sqrt{6} )</td>
</tr>
<tr>
<td>2. ( \frac{\sqrt{125}}{\sqrt{5}} = \sqrt{\frac{125}{5}} = \sqrt{25} = 5 )</td>
</tr>
</tbody>
</table>

> If \( a > 0 \), then \( \sqrt{a^2} = a \)

**Check 1**

Following the patterns shown in Example 1, simplify the following.

| 1. \( \sqrt{3} \times \sqrt{7} \) |
| 2. \( \frac{\sqrt{28}}{\sqrt{7}} \) |

**Problem 1**

Simplify the following.

| 1. \( \sqrt{6} \times \sqrt{5} \) |
| 2. \( \sqrt{2} \times \sqrt{8} \) |
| 3. \( \sqrt{3} \times \sqrt{12} \) |
| 4. \( \sqrt{18} \div \sqrt{6} \) |
| 5. \( \sqrt{12} \div \sqrt{3} \) |
| 6. \( \sqrt{80} \div \sqrt{5} \) |
Example 2
7 \sqrt{2} = \sqrt{49 \times 2}
= \sqrt{49} \times \sqrt{2}
= 7 \sqrt{2}

Check 2
Following the pattern shown in Example 2, simplify the following.

1. $2 \sqrt{2}$
2. $3 \sqrt{7}$
3. $4 \sqrt{5}$
4. $5 \sqrt{3}$

When one of the factors of the radicand is a perfect square, we can make the opposite transformation shown in Example 2, as in Example 3.

In this case we will make the number inside the root sign the smallest natural number possible.

Example 3
1. $\sqrt{18} = \sqrt{9 \times 2}
   = \sqrt{9} \times \sqrt{2}
   = 3 \sqrt{2}$

2. $\sqrt{108} = \sqrt{2^2 \times 3^3}
   = \sqrt{2^2} \times \sqrt{3^3}
   = 2 \times 3 \times \sqrt{3}
   = 6 \sqrt{3}$

As in Example 3, part (2), finding the prime factors of the radicand helps us to find the number which can be moved outside the root sign.

Check 3
Following the patterns shown in Example 3, express the following numbers in the form $a\sqrt{b}$.

1. $\sqrt{28}$
2. $\sqrt{27}$
3. $\sqrt{72}$
4. $\sqrt{96}$

Problem 2
Express the following numbers in the form $a\sqrt{b}$.

1. $\sqrt{99}$
2. $\sqrt{200}$
3. $\sqrt{243}$
Problem 3 Simplify the following.

1 \[ \sqrt[3]{\frac{3}{49}} \]

2 \[ \sqrt{0.64} \]

3 \[ \sqrt{0.0002} \]

Problem 4 Simplify the following.

1 \[ \sqrt{12} \times \sqrt{20} \]

2 \[ \sqrt{14} \times \sqrt{21} \]

3 \[ \sqrt{24} \times \sqrt{6} \]

Check 4 Simplify the following.

1 \[ \sqrt{15} \times \sqrt{10} \]

2 \[ \sqrt{5} \times \sqrt{3} \times \sqrt{5} \times 2 \]

3 \[ \sqrt{3} \times \sqrt{2} \times \sqrt{3} \]

4 \[ \sqrt{6} \]

Problem 4 Simplify the following.

1 \[ \sqrt{18} \times \sqrt{54} \]

2 \[ \sqrt{45} \times \sqrt{85} \]

3 \[ \sqrt{80} \times \sqrt{15} \]

4 \[ 2 \sqrt{3} \times 3 \sqrt{6} \]
Transforming Numbers So That There Is No Root Sign In the Denominator

Let's find the approximate values of \( \frac{1}{\sqrt{2}} \) and \( \frac{\sqrt{2}}{2} \), then compare the two values.

Given a number with a root sign in the denominator, by multiplying both numerator and denominator by the denominator, we can represent the original number without a root sign in the denominator.

**Example 7**

1. \( \frac{\sqrt{2}}{\sqrt{3}} = \frac{\sqrt{2} \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}} = \frac{\sqrt{6}}{3} \)

2. \( \frac{3}{2\sqrt{6}} = \frac{3 \times \sqrt{6}}{2 \times \sqrt{6} \times \sqrt{6}} = \frac{3\sqrt{6}}{4} \)

**Check 5**

Express the following numbers so that there is no root sign in the denominator.

1. \( \frac{2}{\sqrt{5}} \)

2. \( \frac{3}{2\sqrt{3}} \)

**Problem 5**

Express the following numbers so that there is no root sign in the denominator.

1. \( \frac{\sqrt{3}}{\sqrt{2}} \)

2. \( \frac{4}{3\sqrt{2}} \)

3. \( \frac{6}{\sqrt{8}} \)

4. \( \frac{2\sqrt{3}}{\sqrt{6}} \)

**Problem 6**

Simplify the following.

1. \( \sqrt{2} \div \sqrt{7} \)

2. \( \sqrt{27} \div 2\sqrt{6} \)

3. \( 7\sqrt{2} \div (-\sqrt{63}) \)

4. \( \sqrt{80} \div \sqrt{15} \)

Transforming a number so that there is no root sign in the denominator is called "rationalizing the denominator."

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Approximating Square Roots

Using a calculator, find approximate values for the following numbers. Can you find a relation between them?

\[
\begin{align*}
\sqrt{20000} &= \\
\sqrt{200} &= \\
\sqrt{2} &= \\
\sqrt{0.02} &= \\
\sqrt{0.0002} &= \\
\end{align*}
\]

If we move the decimal point of the radicand by two places, the decimal point of the square root moves by one place in the same direction.

This can be explained by the example below.

\[
\sqrt{200} = \sqrt{2 \times 100} = \sqrt{2} \times 10
\]

Problem 7 Which pair of radicals below can be represented with the same number inside the root sign?

(a) \(\sqrt{50}\)  
(b) \(\sqrt{500}\)  
(c) \(\sqrt{50000}\)  
(d) \(\sqrt{0.5}\)

Check 6 Taking \(\sqrt{2.37}\) as 1.539, find the following values.

1. \(\sqrt{23700}\)  
2. \(\sqrt{0.0237}\)

Problem 8 Taking \(\sqrt{3}\) as 1.732 and \(\sqrt{30}\) as 5.477, find the following values.

1. \(\sqrt{300}\)  
2. \(\sqrt{3000}\)  
3. \(\sqrt{0.03}\)  
4. \(\sqrt{0.3}\)  
5. \(\sqrt{12}\)  
6. \(\sqrt{270}\)
Addition and Subtraction of Expressions with Root Signs

1. Can we simplify \( \sqrt{9} + \sqrt{16} \) as \( \sqrt{9+16} \)?

We can add or subtract expressions with root sign like the way we combine similar terms in algebraic expressions.

\[
\begin{align*}
\text{Example 1} & : & 5\sqrt{2} + 3\sqrt{2} & = (5+3)\sqrt{2} = 8\sqrt{2} \\
\text{Example 2} & : & 4\sqrt{5} + 5\sqrt{2} - 3\sqrt{5} - 2\sqrt{2} & = (4-3)\sqrt{5} + (5-2)\sqrt{2} = \sqrt{5} + 3\sqrt{2}
\end{align*}
\]

2. Simplify the following.

\( 6\sqrt{6} + 2\sqrt{6} \)  
\( 4\sqrt{5} - \sqrt{5} \)

Note: The expression \( \sqrt{5} + 3\sqrt{2} \) cannot be simplified further. It indicates a single number.

Check 2: Simplify \( 7\sqrt{5} + 5\sqrt{3} - 2\sqrt{5} - 3\sqrt{3} \)
Problem 1  Simplify the following.

1. \(2\sqrt{3} + 5\sqrt{3}\)
2. \(5\sqrt{7} - 6\sqrt{7}\)
3. \(2\sqrt{10} - 6\sqrt{10} + 7\sqrt{10}\)
4. \(2\sqrt{6} - \sqrt{3} - 8\sqrt{6}\)
5. \(3\sqrt{7} - 3 - 2\sqrt{7} + 2\)
6. \(\sqrt{2} + 2\sqrt{3} - 5\sqrt{2} + \sqrt{3}\)

Even when there are different numbers inside the root sign, by transforming them into the form \(a\sqrt{b}\), in some cases we can calculate the sum or difference.

Example 3  Simplify \(\sqrt{18} + \sqrt{8}\)

Hint  Start the calculation by making the numbers inside the root sign as simple as possible.

Answer  \(\sqrt{18} = \sqrt{3^2 \times 2}\)
          \(\sqrt{8} = \sqrt{2^2 \times 2}\)

Check 3  Simplify the following.

1. \(\sqrt{12} + \sqrt{75}\)
2. \(\sqrt{108} - \sqrt{48}\)

Problem 2  Simplify the following.

1. \(\sqrt{125} - 4\sqrt{5}\)
2. \(\sqrt{18} + \sqrt{50} - \sqrt{32}\)
3. \(4\sqrt{7} + \sqrt{49} - 3\sqrt{28}\)
4. \(3\sqrt{12} - \sqrt{18} - \sqrt{27}\)
5. \(2\sqrt{20} - 3\sqrt{24} + \sqrt{54} - \sqrt{45}\)
Example 4 Simplify the following.

1. \(5\sqrt{2} + \frac{4}{\sqrt{2}}\)
2. \(\frac{\sqrt{24}}{3} - \frac{1}{\sqrt{6}}\)

Hint: Start the calculation by rationalizing the denominators.

Answer:

1. \(5\sqrt{2} + \frac{4}{\sqrt{2}} = 5\sqrt{2} + \frac{4\sqrt{2}}{2} = 5\sqrt{2} + 2\sqrt{2} = 7\sqrt{2}\)

2. \(\frac{\sqrt{24}}{3} - \frac{1}{\sqrt{6}} = \frac{2\sqrt{6}}{3} - \frac{\sqrt{6}}{6} = \frac{4\sqrt{6}}{6} - \frac{\sqrt{6}}{6} = \frac{\sqrt{6}}{2}\)

Problem 3 Simplify the following.

1. \(\sqrt{50} - \frac{6}{\sqrt{2}}\)
2. \(\frac{\sqrt{5}}{2} + \frac{1}{\sqrt{5}}\)
3. \(2\sqrt{60} + \frac{5}{\sqrt{3}}\)
4. \(\sqrt{3} + \frac{\sqrt{27}}{\sqrt{3}} - \frac{12}{\sqrt{3}}\)

We can use the distributive law to evaluate expressions including root signs.

Example 5 \(\sqrt{3} (\sqrt{6} + 2) = \sqrt{3} \times \sqrt{6} + \sqrt{3} \times 2 = \sqrt{3} \times (\sqrt{3} \times \sqrt{2}) + 2\sqrt{3} = 3\sqrt{2} + 2\sqrt{3}\)

Problem 4 Simplify the following.

1. \(\sqrt{2} (1 + 3\sqrt{2})\)
2. \(2\sqrt{3} (\sqrt{12} - \sqrt{6})\)
3. \(\sqrt{5} (2\sqrt{35} - \sqrt{15})\)
4. \(\sqrt{2} (-\sqrt{10} + \sqrt{14})\)
Basic Exercises

Transforming numbers with root signs

1. Express the following numbers in the form $\sqrt{a}$.
   - $3\sqrt{2}$
   - $4\sqrt{6}$
   - $6\sqrt{5}$

2. Express the following numbers in the form $a\sqrt{b}$.
   - $\sqrt{24}$
   - $48\sqrt{3}$
   - $\sqrt{50}$

3. Express the following numbers with no root sign in the denominator.
   - $\frac{3}{\sqrt{2}}$
   - $\frac{\sqrt{5}}{\sqrt{7}}$
   - $\frac{9}{2\sqrt{3}}$

Multiplication and division of expressions including root signs

4. Simplify the following.
   - $\sqrt{14} \times \sqrt{5}$
   - $\sqrt{27} \times \sqrt{8}$
   - $\sqrt{6} \times \sqrt{42}$
   - $\sqrt{32} \times \sqrt{2}$
   - $\sqrt{75} : \sqrt{3}$
   - $6\sqrt{5} : \sqrt{6}$

Addition and subtraction of expressions including root signs

5. Simplify the following.
   - $8\sqrt{2} + 4\sqrt{2}$
   - $8\sqrt{3} - 5\sqrt{3}$
   - $4\sqrt{5} + 3\sqrt{7} - 2\sqrt{5} + \sqrt{7}$
   - $4\sqrt{3} - 3\sqrt{6} - \sqrt{3} + 2\sqrt{6}$
   - $\sqrt{32} + \sqrt{8}$
   - $\sqrt{45} - \sqrt{20}$
   - $\sqrt{5} (\sqrt{10} - 1)$
Chapter Summary Problems A

1. Use inequality signs to compare the following numbers.
   ① $3, \sqrt{10}$  
   ② $-6, -\sqrt{38}, -\sqrt{35}$

2. Express the following numbers so that there is no root sign in the denominator.
   ① $\frac{2}{\sqrt{7}}$  
   ② $\frac{7}{3\sqrt{2}}$  
   ③ $\frac{4}{\sqrt{8}}$  
   ④ $\frac{3\sqrt{2}}{\sqrt{6}}$

3. Are the following statements correct? If not, correct the underlined number.
   ① The square root of 64 is $8$.  
   ② $\sqrt{(-6)^2}$ is equal to $-6$.  
   ③ $\sqrt{16}$ is $\pm4$.  
   ④ $\sqrt{7} \times \sqrt{7}$ is equal to $7$.  
   ⑤ $\sqrt{0.9}$ is equal to 0.3.  
   ⑥ $\sqrt{16} - \sqrt{9}$ is equal to $\sqrt{7}$.

4. Simplify the following.
   ① $\sqrt{7} \times \sqrt{56}$  
   ③ $\sqrt{14} \div \sqrt{42}$  
   ⑤ $2\sqrt{7} + 5\sqrt{7}$  
   ⑦ $\sqrt{18} - \sqrt{2}$  
   ⑨ $\frac{15}{\sqrt{5}} \div \sqrt{\frac{20}{4}}$  
   ② $\sqrt[3]{80} \times \sqrt[3]{12}$  
   ④ $10 \div \sqrt{15}$  
   ⑥ $5\sqrt{3} + \sqrt{3} - 3\sqrt{5} + 4\sqrt{3}$  
   ⑧ $\sqrt{112} - \sqrt{28} + \sqrt{7}$  
   ⑩ $\sqrt{3} (\sqrt{12} + 2\sqrt{18})$

5. A square prism has a volume of 500 cm$^3$, and height of 10 cm. How many centimeters long is one side of the base? Give your answer correct to one decimal place.
Chapter Summary Problems  B

1 Simplify the following.
   \[ \frac{\sqrt{75}}{5\sqrt{2} \times \sqrt{6}} \]
   \[ \frac{2\sqrt{5} - \sqrt{15} \times \sqrt{3}}{2\sqrt{2} - 2\sqrt{6} \times \sqrt{12}} \]

2 Express the following numbers so that there is no root sign in the denominator.
   \[ \frac{\sqrt{15}}{\sqrt{7} \sqrt{3}} \]
   \[ \frac{\sqrt{5} + \sqrt{6}}{\sqrt{3}} \]

3 Answer the following questions where \( a \) is a positive integer.
   \[ \text{Find all values of } a \text{ such that } 3.7 < \sqrt{a} < 4. \]
   \[ \text{Find all values of } a \text{ for which } \sqrt{14} - a \text{ is a natural number.} \]

4 By what smallest possible number should we divide 216 so that the quotient has no remainder and is a perfect square of a natural number?

Let's investigate!

Finding the prime factors of an integer will help us find all the factors of the integer. For example, the prime factors of 20 is given by:
\[ 20 = 2^2 \times 5 \]
Which means that 20 has the following six factors:
\[
\begin{array}{c|ccc}
1 & 2^2 = 4 & \text{Prime factors} & \text{Products of prime factors} \\
2 & 2 \times 5 = 10 & & \\
5 & 2^2 \times 5 = 20 & & \\
\end{array}
\]

Look at the prime factors of numbers, to find all of their factors.

Using this, see if you can find a way of calculating the number of different factors of a number, and ways of finding the greatest common divisor and least common multiple of two integers.
Using the following method, we can cross out the non-prime numbers below to find all prime numbers up to 100.

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1. Cross out 1, which is not a prime.
2. Circle 2, then cross out all following multiples of 2.
3. Circle the smallest remaining number, 3, then cross out all following multiples of 3.
4. Repeat this process, circling the smallest remaining number, then crossing out all following multiples of this number.
5. The circled numbers are the primes.

Try carrying out this procedure to find all prime numbers up to 100. To do this, what is the largest number whose multiples we need to cross out?

This method was discovered by the Greek mathematician Eratosthenes, some 2200 years ago, and for this reason it is known as the "Sieve of Eratosthenes."
Numbers Which Cannot Be Expressed as Fractions

As we wrote on page 9, the number $\sqrt{5}$ cannot be expressed as a fraction. In other words, it cannot be written as $\frac{a}{b}$, where $a$ is an integer, and $b$ is a nonzero integer.

Numbers which cannot be expressed as fractions in this way are called irrational numbers (or irrationals).

On the other hand, numbers which can be expressed as fractions are called rational numbers (or rationals).

For any natural number $n$, except in cases where $n$ is a perfect square, such as 9 and 16, $\sqrt{n}$ is always an irrational number.

The number $\pi$ is also irrational.

Thinking about why $\sqrt{5}$ is irrational

As we found in Problem 3 on page 8, $2 < \sqrt{5} < 3$, and therefore $\sqrt{5}$ is not an integer. Then suppose that:

$$\sqrt{5} \text{ can be expressed as } \frac{a}{b}$$

Further suppose that the fraction $\frac{a}{b}$ is in lowest terms, that is, $a$ and $b$ have no common factors except 1.

Squaring both sides of $\sqrt{5} = \frac{a}{b}$ gives:

$$5 = \left(\frac{a}{b}\right)^2$$

Since the numerator and denominator of $\frac{a}{b}$ have no factors in common except 1, the same must be true of $\left(\frac{a}{b}\right)^2 = \frac{a}{b} \times \frac{a}{b}$. Therefore, $\left(\frac{a}{b}\right)^2$ cannot be equal to the integer 5. In other words, (2) does not hold. This must have happened because our original assumption (1) was false. Therefore we can see that $\sqrt{5}$ cannot be expressed as a fraction.

The panel on the right summarizes the types of number we have learned about up to now.

---

26 1 — Square Roots

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Answer
Non-terminating Decimals

If we express an irrational number as a decimal, the decimal part continues without ever coming to an end; it is non-terminating.

If we express a rational as a decimal, in some cases, such as \( \frac{3}{4} = 0.75 \) the decimal terminates after a certain number of decimal places. But in other cases, such as \( \frac{1}{3} = 0.3333... \), we again have a non-terminating decimal. Let's look at the properties of these non-terminating decimals.

If, for example, we express \( \frac{9}{7} \) as a decimal we have:

\[
\frac{9}{7} = 1.285714285714... 
\]

And the sequence 285714 in pink is repeated endlessly. This kind of non-terminating decimal we call a recurring decimal.

To show a recurring decimal, we put a dot over the repeating digit, or over the beginning and end of the repeating sequence.

We can convert a recurring decimal to a fraction using the fact that \( \frac{1}{9} = 0.1 \), \( \frac{1}{99} = 0.01 \), \( \frac{1}{999} = 0.001 \) and so on.

**Example 1**

\[
\frac{1}{3} = 0.3, \quad \frac{9}{7} = 1.285714 
\]

We can convert a recurring decimal to a fraction using the fact that \( \frac{1}{9} = 0.1 \), \( \frac{1}{99} = 0.01 \), \( \frac{1}{999} = 0.001 \) and so on.

**Example 2**

\[
0.2\dot{5} = 0.0\dot{1} \times 25 = \frac{1}{99} \times 25 = \frac{25}{99} 
\]

**Problem 1** Convert 0.1 and 0.12\(\dot{3} \) to fractions.

If we express a rational number as a decimal the result is either terminating or recurring. Conversely, since we can express a terminating decimal as a fraction, it is a rational number, and since we can also express a recurring decimal as a fraction it too is a rational number too. Therefore, an irrational number is a non-terminating decimal which does not recur.
Point C lies on line segment AB. Figures 1 to 3 show semicircles having diameters AC and CB while changing the position of point C each time. Compare the length of the red line in cases 1 to 3.
The following diagram shows case 1 on the previous page, with a semicircle having diameter AB also drawn. Compare the lengths of the blue and red lines.

In the above diagrams, if AC = \( a \), and CB = \( b \), write expressions for the lengths of the blue and red lines.
Multiplication and Division of Polynomials and Monomials

In the problem on the previous page, an expression for the length of the red line suggested by the hint $\frac{\pi}{2}$ is as follows:

$$a \times \pi \div 2 + b \times \pi \div 2 = \frac{\pi a + \pi b}{2}$$

On the other hand, since the blue line is a semicircular arc with diameter $AB$, its length is given by the following expression:

$$(a + b) \times \pi \div 2 = \frac{\pi (a + b)}{2} = \frac{\pi a + \pi b}{2}$$

Therefore, the length of the red line is equal to the length of a semicircular arc with diameter $AB$.

Thus our algebraic manipulation shows that the length of the red line is the same wherever the point $C$ is on the line segment $AB$.

In calculating $\pi (a + b)$ we used the distributive law. We can multiply monomials and polynomials by using the distributive law.

$$\pi (a + b) = \pi a + \pi b$$

**Example 1**

1. $2a(3a - 5b) = 2a \times 3a - 2a \times 5b$
   
   $= 6a^2 - 10ab$

2. $(x - 2y + 5) \times (-3x)$
   
   $= x \times (-3x) - 2y \times (-3x) + 5 \times (-3x)$

   $= -3x^2 + 6xy - 15x$

**Check 1**

1. $4a(a + 3b)$

2. $(2x - 7y) \times (-5x)$

**Problem 1**

1. $-b(5a - b)$

2. $\frac{2}{3} x(3x - 6)$

3. $2a(a - b - c)$

4. $(3x + 2y - 1) \times (-6x)$

30 2 — Polynomials
Let's look at dividing a polynomial by a monomial.

**Example 2**

1. \((4xy^2 + 6x^2y) ÷ 2x = (4xy^2 + 6x^2y) \times \frac{1}{2x}\)
   \[= \frac{4xy^2}{2x} + \frac{6x^2y}{2x}\]
   \[= 2y^2 + 3xy\]

2. \((4a^2 + ab) ÷ \frac{1}{2}a = (4a^2 + ab) \times \frac{2}{a}\)
   \[= \frac{4a^2 \times 2}{a} + \frac{ab \times 2}{a}\]
   \[= 8a + 2b\]

**Check 2** Simplify the following.

1. \((2x^2y - 3xy^2) ÷ y\)
2. \((6ab - 2ab^2) ÷ \frac{2}{3}a\)

**Problem 2** Simplify the following.

1. \((8a^2b + 2b) ÷ (-2b)\)
2. \((6a^2b - 9ab^2) ÷ 3ab\)
3. \((x^2y - x) ÷ x\)
4. \((12a^2b - 8ab) ÷ \frac{4}{5}ab\)

Let's look at some slightly more complicated examples.

**Example 3**

\[2x(x + 3) + x(2 - x) = 2x^2 + 6x + 2x - x^2\]
\[= x^2 + 8x\]

**Problem 3** Simplify the following.

1. \(2x(x - 4) + 3x(x + 5)\)
2. \(4a(a - 3) - 2a(3a - 6)\)
3. \(-3x(5 - x) - 4x(1 + x)\)
4. \(a(a + 2b) - \frac{2}{3}a(a + 9b)\)
2 Multiplication of Polynomials

Trigger

A rectangle has width $a + b$ and length $c + d$.

Let's find ways of writing an expression for the area of this rectangle.

Let's consider multiplying two polynomials together.

In the evaluation of $(a + b)(c + d)$, if we let $c + d = M$, this takes the form of multiplying a monomial by a polynomial, which we can evaluate using the distributive law.

$$\begin{align*}
(a + b)(c + d) &= (a + b)M \\
&= aM + bM \\
&= a(c + d) + b(c + d) \\
&= ac + ad + bc + bd
\end{align*}$$

Replace $c + d$ by $M$

Use the distributive law to multiply out

Change $M$ back to $c + d$ law to multiply out

Use the distributive law to multiply out

**Problem 1**

Letting $a + b = M$, evaluate $(a + b)(c + d)$.

To evaluate $(a + b)(c + d)$, we need to form the sum of the following combinations.

We call this process, taking an expression in the form of a product of monomials or polynomials and multiplying out to make a sum of terms "expanding" the original expression.
Check 1 Expand the following expressions.
1  \((x + 6)(y + 2)\)  
2  \((a - 3)(b + 2)\) 

If there are similar terms in the expansion, we simplify by collecting these together.

Example 2
\((3x + 2)(x - 4)\)
= \(3x^2 - 12x + 2x - 8\)
= \(3x^2 - 10x - 8\)

Check 2 Expand the following expressions.
1  \((x + 7)(x + 4)\)  
2  \((4x - 3)(2x + 1)\)

Problem 2 Expand the following expressions.
1  \((a - b)(c - d)\)  
2  \((2x + 1)(y - 7)\)  
3  \((x + 2)(x + 4)\)  
4  \((x - 2)(x - 3)\)  
5  \((2a + b)(a + 3b)\)  
6  \((4x - 1)(3x - 2)\)

Let's look at an example where there are more terms inside the parentheses.

Example 3
\((a + 3)(a + 2b - 4)\)
= \(a(a + 2b - 4) + 3(a + 2b - 4)\)
= \(a^2 + 2ab - 4a + 3a + 6b - 12\)
= \(a^2 + 2ab + 7b - 12\)

Think of \(a + 2b - 4\) as a single term.

Problem 3 Expand the following expressions.
1  \((a + 1)(a - b + 2)\)  
2  \((2x + y - 1)(5x - 3y)\)
Special Products

Product of $x + a$ and $x + b$

If we expand the following expressions into the form $x^2 + bx + a$, what will be the values of $b$ and $a$?

1. $(x + 4)(x + 3)$
2. $(x + 4)(x - 3)$

**Formula 1**

$$(x + a)(x + b) = x^2 + (a + b)x + ab$$

**Problem 1**

Expand the left hand side of Formula 1 to check that this formula holds.

**Example 1**

Expand $(x + 2)(x + 7)$.

In Formula 1, when $a$ is 2 and $b$ is 7, we have:

$$(x + 2)(x + 7) = x^2 + (2 + 7)x + 2 \times 7$$
$$= x^2 + 9x + 14$$

**Check 1**

Expand $(x + 3)(x + 6)$. 

34 --- Polynomials
\[(x + 3)(x - 4) = (x + 3)[x + (-4)] = x^2 + [3 + (-4)]x + 3 \times (-4) = x^2 - x - 12\]

Check 2: Expand \((x + 1)(x - 3)\):

Problem 2: Expand the following expressions.

1. \((x + 1)(x + 2)\)
2. \((x + 5)(x - 2)\)
3. \((x - 3)(x - 4)\)
4. \((y + 3)(y + 5)\)
5. \((a - 7)(a - 3)\)
6. \((x - 6)(x + 5)\)
7. \((x - 2)(x + 4)\)
8. \((y - \frac{2}{3})(y + \frac{1}{3})\)

Squares of Sums and Differences

We can expand \((x + a)^2\), using Formula 1, as follows.

\[\begin{align*}
(x + a)^2 &= (x + a)(x + a) \\
&= x^2 + (a + a)x + a \times a \\
&= x^2 + 2ax + a^2
\end{align*}\]

Similarly, we can expand \((x - a)^2\) as follows.

\[\begin{align*}
(x - a)^2 &= (x - a)(x - a) \\
&= x^2 + [(-a) + (-a)]x + (-a) \times (-a) \\
&= x^2 - 2ax + a^2
\end{align*}\]

Formula 2: \((x + a)^2 = x^2 + 2ax + a^2\)

Formula 3: \((x - a)^2 = x^2 - 2ax + a^2\)

Note: We can also derive Formula 3 by substituting \(-a\) for \(a\) in Formula 2.
Example 3 Expand \((x + 3)^2\).

In Formula 2, \(a\) is 3, and therefore:
\[
(x + 3)^2 = x^2 + 2 \times 3 \times x + 3^2 = x^2 + 6x + 9
\]

Example 4

\[(x - 8)^2 = x^2 - 2 \times 8 \times x + 8^2 = x^2 - 16x + 64\]

Check 3 Expand the following expressions.

1. \((x + 6)^2\)
2. \((y - 5)^2\)

Problem 3 Expand the following expressions.

1. \((a + 9)^2\)
2. \((a - 7)^2\)
3. \((a - b)^2\)
4. \((x + \frac{1}{3})^2\)
5. \((2 - x)^2\)
6. \((-x + 1)^2\)

The Product of the Sum and the Difference

Trigger By substituting \(-a\) for \(b\) in Formula 1, we obtain the following formula.

Formula 4 \((x + a)(x - a) = x^2 - a^2\)

Example 5

\[(x + 6)(x - 6) = x^2 - 6^2 = x^2 - 36\]

Check 4 Expand \((x + 3)(x - 3)\).
Problem 4
Expand the following expressions.

1. \((a + b)(a - b)\)
2. \((x + 5)(x - 5)\)
3. \((x - 8)(x + 8)\)
4. \((2 + x)(2 - x)\)
5. \((y + \frac{1}{7})(y - \frac{1}{7})\)
6. \((a + 4)(4 - a)\)

We can summarize the formulas we have learned so far with the following list.

- Multiplication formulas

1. \((x + a)(x + b) = x^2 + (a + b)x + ab\)
2. \((x + a)^2 = x^2 + 2ax + a^2\)
3. \((x - a)^2 = x^2 - 2ax + a^2\)
4. \((x + a)(x - a) = x^2 - a^2\)

Problem 5
Expand the following expressions.

1. \((x - 4)^2\)
2. \((x - 6)(x + 4)\)
3. \((x + 7)(x - 7)\)
4. \((x + y)^2\)
5. \((x - 6)(x + 2)\)
6. \((8 + a)^2\)
7. \((a + 2)(a - 4)\)
8. \((9 - x)(9 + x)\)

Let’s try!

Expand the following expressions.

1. \((x - \frac{1}{2})^2\)
2. \((4 + x)(-5 + x)\)
3. \((a + \frac{1}{3})(a + \frac{2}{3})\)
4. \((x - \frac{1}{4})(x + \frac{1}{4})\)
5. \((5 + x)(7 + x)\)
6. \((1 + x)(x - 1)\)
More Expansions

Let's look at expanding slightly more complicated expressions.

Let's expand \((2x+1)(2x+3)\)

To expand \((2x+1)(2x+3)\) we can use Formula 1 with the following calculation.

\[
(2x + 1)(2x + 3) = (2x)^2 + (1+3)x + 2x + 1 \times 3 = 4x^2 + 8x + 3
\]

\[
(2x + 1)(2x + 3) = (2x)^2 + (1+3)x + 2x + 1 \times 3
\]

\[
(A + 1)(A + 3) = A^2 + (1+3)A + 1 \times 3
\]

Problem 6

Expand the following expressions.

1. \((3x - 4)(3x - 2)\)
2. \((-4a + 3)(-4a - 6)\)
3. \((2x - 5y)(2x + y)\)
4. \((\frac{1}{2}x + 5)(\frac{1}{2}x - 3)\)

Example 6

Expand \((2x - 3y)^2\).

Hint: We can use Formula 3 for the expansion as follows.

\[
(2x - 3y)^2 = (2x)^2 - 2 \times 3y \times 2x + (3y)^2
\]

\[
(x - a)^2 = x^2 - 2ax + a^2
\]

Answer

\[
(2x - 3y)^2 = (2x)^2 - 2 \times 3y \times 2x + (3y)^2
\]

\[
= 4x^2 - 12xy + 9y^2
\]
Problem 7 Expand the following expressions.

1. \((5x + 2)^2\)
2. \((a + 2b)^2\)
3. \((3a - 5b)^2\)
4. \(\left(\frac{1}{2}a - 4b\right)^2\)

Example 7

\((5x + 4)(5x - 4) = (5x)^2 - 4^2\)
\[= 25x^2 - 16\]

Problem 8 Expand the following expressions.

1. \((5x + 7)(5x - 7)\)
2. \((3 + 5a)(3 - 5a)\)
3. \((7x - 4y)(7x + 4y)\)
4. \((-2x + 3y)(-2x - 3y)\)

Now let's try evaluating expressions involving expansion and addition and subtraction.

Example 8 Evaluate \(2(x + 5)^2 - (x + 3)(x - 3)\)

Answer

\[
2(x + 5)^2 - (x + 3)(x - 3) = 2(x^2 + 10x + 25) - (x^2 - 9) = 2x^2 + 20x + 50 - x^2 + 9 = x^2 + 20x + 59
\]

Problem 9 Evaluate the following expressions.

1. \((x - 2)^2 + (x + 4)(x + 1)\)
2. \(2(x + 1)(x - 1) + (x - 3)(x + 2)\)
3. \(4(a + 1)^2 - (2a - 1)^2\)
4. \((x - 3)(x + 5) - (3x - 1)(3x + 1)\)
Using the multiplication formula, let's evaluate expressions including root signs.

\[ \sqrt{2} + 1 \cdot (\sqrt{2} + 3) = (\sqrt{2})^2 + (1 + 3)\sqrt{2} + 1 \times 3 \]
\[ = 2 + 4\sqrt{2} + 3 \]
\[ = 5 + 4\sqrt{2} \]

\[ (\sqrt{2} + 1) \cdot (\sqrt{2} + 3) = (\sqrt{2})^2 + (1 + 3)\sqrt{2} + 1 \times 3 \]
\[ (x + a) \cdot (x + b) = x^2 + (a + b)x + ab \]

**Problem 10**
Evaluate the following expressions.

1. \( (\sqrt{2} - \sqrt{3})^2 \)
2. \( (\sqrt{3} + 3\sqrt{5})^2 \)
3. \( (\sqrt{2} - 2)(\sqrt{2} + 4) \)
4. \( (\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5}) \)

**Eliminating Root Signs from the Denominator of an Expression**

Using Formula 4 for multiplication, we can transform expressions like \( \frac{1}{\sqrt{3} + \sqrt{2}} \) so that there is no root sign in the denominator.

\[ \frac{1 \times (\sqrt{3} - \sqrt{2})}{(\sqrt{3} + \sqrt{2})(\sqrt{3} - \sqrt{2})} = \frac{\sqrt{3} - \sqrt{2}}{(\sqrt{3})^2 - (\sqrt{2})^2} \]
\[ = \frac{\sqrt{3} - \sqrt{2}}{3 - 2} \]
\[ = \sqrt{3} - \sqrt{2} \]

**Problem 1**
Solve the following problems for the expression \( \frac{2}{\sqrt{5} + 1} \).

1. Transform the expression so that there is no root sign in the denominator.
2. If \( \sqrt{5} = 2.236 \), find the value of \( \frac{2}{\sqrt{5} + 1} \).
Basic Exercises

1. Multiplication and division of polynomials and monomials
   Evaluate the following expressions.
   1. \(3a(2a - 3b)\)  
   2. \((6x^2y - 12y) ÷ 6y\)

2. Expanding expressions
   Expand the following expressions.
   1. \((x + 3)(2x - 4)\)  
   2. \((a - 4)(a - 2b + 3)\)

3. Multiplication formula
   Expand the following expressions.
   1. \((a + 5)(a + 1)\)  
   2. \((x - 3)(x - 8)\)  
   3. \((a - 3)(a - 5)\)  
   4. \((x + 7)(x - 6)\)  
   5. \((x - 2)(x + 7)\)  
   6. \((x + 5)^2\)  
   7. \((y - 3)^2\)  
   8. \((a + 2)(a - 2)\)

4. Expand the following expressions.
   1. \((3x - 5)(3x + 2)\)  
   2. \((x + 4y)(x - 5y)\)  
   3. \((2a - 7)^2\)  
   4. \((4x - 3)(4x + 3)\)

5. Evaluating expressions including root signs
   Evaluate the following expressions.
   1. \((\sqrt{2} + \sqrt{3})^2\)
   2. \((\sqrt{5} + 2)(\sqrt{5} - 2)\)
   3. \((\sqrt{3} - 2)(\sqrt{3} - 4)\)
Rearrange the rectangles and squares below to form a different rectangle. What will be the lengths of the sides of the rectangle formed?

Using the multiplication formula in reverse we see that:

\[ x^2 + 3x + 2 = (x + 1)(x + 2) \]

This shows that \( x^2 + 3x + 2 \) is the product of \((x + 1)\) and \((x + 2)\). In this case we say that \((x + 1)\) and \((x + 2)\) are factors of \( x^2 + 3x + 2 \).

**Example 1**

1. In the expression \( 2ab \), \( 2, a, b, 2 \) \( a \), and so on are factors.

2. Since \( x^2 + 2x = x(x + 2) \), \( x \) and \((x + 2)\) are factors of \( x^2 + 2x \).

When we express a polynomial as a product of factors, we call this a factorization, and call the process factoring.
Common Factors

When the terms of a polynomial have a common factor, we can move the common factor outside parentheses, to give the factors of the polynomial.

\[ m \ a + m \ b + m \ c = m \ (a + b + c) \]

**Example 2** Factorize \( x^2 + 2xy \).

\[
x^2 = x \times x \\
2xy = 2 \times x \times y
\]

Therefore the two terms have \( x \) as a common factor. Thus we can obtain the factorization:

\[ x^2 + 2xy = x \ (x + 2y) \]

**Example 3**

\[
3ax - 6ay \\
= 3a(x - 2y)
\]

\[ 3ax = 3 \times a \times x \]
\[ 6ay = 2 \times 3 \times a \times y \]

**Note** We could also factorize \( 3ax - 6ay \) to give \( a(3x - 6y) \), but this leaves a common factor of 3 for the terms inside the parentheses. In cases like this, we try to factor out the greatest common factor, as shown in Example 3 above.

**Check 1** Factorize the following expressions.

1. \( ax - bx \)
2. \( 2x^2y - 4x \)

**Problem 1** Factorize the following expressions.

1. \( 6mx - 2nx \)
2. \( 5x^2 - 10xy \)
3. \( xy^2 - x^2y \)
4. \( 4a^2b - 6ab^2 - 10ab \)

4. We need to find factors common to all three terms.
Factorization Using Formulas

We can use the multiplication formula in reverse for factorization.

Formula 1′: \( x^2 + (a + b)x + ab = (x + a)(x + b) \)

Example 1
Let’s factorize \( x^2 + 5x + 6 \).
Comparing \( x^2 + 5x + 6 \) with Formula 1′, we see that
\[ a + b = 5, \quad ab = 6 \]
Therefore we have to find numbers \( a \) and \( b \) whose sum is 5 and whose product is 6. Of the combinations of numbers with a product of 6, the one with a sum of 5 is the pair of 2 and 3, and therefore:
\[ x^2 + 5x + 6 = (x + 2)(x + 3) \]

Example 2
Factorize \( x^2 + x - 6 \).
Hint:
Of the combinations of two numbers with a product of -6, find the one with a sum of 1.
Answer:
\[ x^2 + x - 6 = (x - 2)(x + 3) \]
Check 2  Factorize \( x^2 - 2x - 15 \).

<table>
<thead>
<tr>
<th>Product is -15</th>
<th>Sum is -2?</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Problem 2  Factorize the following expressions.
1. \( x^2 - 2x - 8 \)
2. \( x^2 + 3x - 10 \)
3. \( a^2 - 7a - 8 \)

Note  Some quadratic polynomials, such as \( x^2 + 2x + 3 \) cannot be factored.

Problem 3  Factorize the following expressions.
1. \( x^2 + 10x + 9 \)
2. \( y^2 + 5y - 36 \)
3. \( x^2 - 3x - 28 \)
4. \( x^2 - 16x + 28 \)

Formula 2'  \( x^2 + 2ax + a^2 = (x + a)^2 \)
Formula 3'  \( x^2 - 2ax + a^2 = (x - a)^2 \)

Let's factorize \( x^2 + 10x + 25 \).

\[
10 = 2 \times 5, \quad 25 = 5^2
\]

And therefore, using Formula 2' we have:

\[
x^2 + 10x + 25 = x^2 + 2 \times 5 \times x + 5^2
\]

\[
= (x + 5)^2
\]

Check 3  Factorize \( x^2 + 6x + 9 \).

Problem 4  Factorize the following expressions.
1. \( x^2 + 8x + 16 \)
2. \( a^2 + 18a + 81 \)
3. \( x^2 - 2x + 1 \)
4. \( a^2 - 12a + 36 \)

Problem 5  Factorize \( x^2 + 10x + 25 \), using Formula 1'.
Formula 4’ \[ x^2 - a^2 = (x + a)(x - a) \]

Example 4 \[ x^2 - 25 = x^2 - 5^2 \]
\[ = (x + 5)(x - 5) \]

Check 4 Factorize \( x^2 - 36 \).

Problem 6 Factorize the following expressions.
1. \( a^2 - 4 \)
2. \( x^2 - 81 \)
3. \( x^2 - 100 \)
4. \( 16 - y^2 \)

Problem 7 Using Formula 1’, factorize \( x^2 - 25 \).

We can summarize the formulas we have learned so far as follows.

<table>
<thead>
<tr>
<th>Factorization formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( x^2 + (a + b)x + ab = (x + a)(x + b) )</td>
</tr>
<tr>
<td>2. ( x^2 + 2ax + a^2 = (x + a)^2 )</td>
</tr>
<tr>
<td>3. ( x^2 - 2ax + a^2 = (x - a)^2 )</td>
</tr>
<tr>
<td>4. ( x^2 - a^2 = (x + a)(x - a) )</td>
</tr>
</tbody>
</table>

Problem 8 Factorize the following expressions.
1. \( x^2 - 3x + 2 \)
2. \( x^2 - 64 \)
3. \( a^2 + 14a + 49 \)
4. \( x^2 - 5x - 24 \)
5. \( x^2 + 13x + 36 \)
6. \( y^2 - 4y + 4 \)
7. \( 9 - a^2 \)
8. \( x^2 + 7x - 18 \)
More Factorizations

Let's try factorizing more expressions.

Example 5

\[ x^2 - 6xy + 9y^2 \]
\[ = x^2 - 2 \times 3y \times x + (3y)^2 \]
\[ = (x - 3y)^2 \]
\[ x^2 - 2 \times 3y \times x + (3y)^2 \]
\[ = x^2 - 2ax + a^2 \]

Problem 9

Factorize the following expressions.

1. \[ 9x^2 + 6x + 1 \]
2. \[ 4a^2 - 20a + 25 \]
3. \[ x^2 - 20xy + 100y^2 \]
4. \[ 4x^2 + 12xy + 9y^2 \]

Example 6

\[ 4x^2 - 9y^2 = (2x)^2 - (3y)^2 \]
\[ = (2x + 3y)(2x - 3y) \]

Problem 10

Factorize the following expressions.

1. \[ x^2 - 25y^2 \]
2. \[ 9a^2 - b^2 \]
3. \[ 4a^2 - 25b^2 \]
4. \[ x^2 - \frac{y^2}{4} \]

Example 7

\[ 2x^2 + 4x - 16 \]
\[ = 2(x^2 + 2x - 8) \]
\[ = 2(x - 2)(x + 4) \]

Pull out the common factor, then factorize inside the brackets.

Check 5

Factorize \[ 2x^2 + 16x + 24 \].

What is the common factor?

Problem 11

Factorize the following expressions.

1. \[ 3x^2 + 18x - 48 \]
2. \[ -3y^2 + 18y - 27 \]
3. \[ 2x^2y - 8xy + 6y \]
4. \[ 27x^2y - 12yz^2 \]

p. 54 1 Using Algebraic Variables (Development) 2 — Factorization
Applications

Let’s look at some numerical calculations which we can do by evaluating expressions.

The diagram on the right shows two squares. Find the area of the colored part.

We can calculate $35^2 - 15^2$ by using Formula 4' as follows:

$$35^2 - 15^2 = (35 + 15)(35 - 15)$$

$$= 1000$$

Problem 1 Use your ingenuity to calculate the following.

1. $47 \times 53$
2. $99^2$

Perhaps we can use a multiplication formula?

If $x = \sqrt{3} + 2$ and $y = \sqrt{3} - 2$, find the value of $x^2 - xy$.

**Answer**

$$x^2 - xy = (\sqrt{3} + 2)^2 - (\sqrt{3} + 2)(\sqrt{3} - 2)$$

$$= (3 + 4\sqrt{3} + 4) - (3 - 4)$$

$$= (7 + 4\sqrt{3}) - (-1)$$

$$= 8 + 4\sqrt{3}$$

Answer $8 + 4\sqrt{3}$

Problem 2 In Example 2, substitute the values of $x$ and $y$ into the factors of $x^2 - xy$ to find the value of the expression.

Check 1 If $x = \sqrt{3} + 2$, find the value of $x^2 - 4x + 3$.

Problem 3 If $x = \sqrt{2} + \sqrt{3}$, $y = \sqrt{2} - \sqrt{3}$, find the values of the following expressions.

1. $(x + y)^2$
2. $x^2 - y^2$

48 2 — Polynomials
Suppose we take two successive odd numbers, such as 1 and 3. If we add 1 to the product of these two numbers, what result can we expect?

\[
1 \times 3 + 1 = 4 \\
3 \times 5 + 1 = 
\]

Since an odd number is one which is not divisible by 2, it is equal to a multiple of 2, plus 1. On the other hand, an even number is divisible by 2, and so is a multiple of 2.

If we add 1 to the product of two successive odd numbers, the result is a multiple of 4. Prove this.

We can represent two successive odd numbers in terms of an integer \( n \) as follows:

\[
2n - 1, \quad 2n + 1
\]

The result of adding 1 to the product of these two numbers is:

\[
(2n - 1)(2n + 1) + 1 = 4n^2 - 1 + 1 = 4n^2
\]

Therefore the result is a multiple of 4.

Given three consecutive integers. Adding the middle integer to the product of the three integers equals the cube of the middle integer. Prove this, representing the middle integer as \( n \).

Given two successive odd numbers. If we subtract the square of the lesser number from the square of the greater number, what number would we expect the result to be a multiple of? Prove that your answer is true.
Let's look at some problems in calculating areas which we can do by evaluating expressions.

Example 4

Around the periphery of a circular plot of land of radius \( r \) meters is a track of width \( a \) meters.

If the area of this track is \( S \) square meters, and the circumferential length of a circle along the center of the track is \( \ell \) meters, then:

\[
S = a \ell
\]

Prove that this holds.

How can we express the radius of the circular outer edge of the track, and the radius of the circle along the center of the track.

Answer

\[
S = \pi (r + a)^2 - \pi r^2
\]

\[
= \pi (r^2 + 2ar + a^2) - \pi r^2
\]

\[
= 2\pi ar + \pi a^2
\]

(1)

Since the radius of the circle passing along the center of the track is \( r + \frac{a}{2} \) meters, we have:

\[
\ell = 2\pi \left( r + \frac{a}{2} \right)
\]

\[
= 2\pi r + \pi a
\]

And therefore:

\[
a \ell = a \left( 2\pi r + \pi a \right)
\]

\[
= 2\pi ar + \pi a^2
\]

(2)

From (1) and (2), \( S = a \ell \)

Problem 6

Around the outer edge of a square plot of land of side \( p \) meters is a track of width \( a \) meters.

If the area of this track is \( S \) square meters, and the length of a path passing along the center of the track is \( \ell \) meters, then:

\[
S = a \ell
\]

Prove that this holds.
Basic Exercises

Factorization

1. Factorize the following expressions.
   \( ab - 5b \)
   \( 4x^2y + 2xy^2 \)

2. Factorize the following expressions.
   \( a^2 + 7a + 12 \)
   \( x^2 - 12x + 27 \)
   \( x^2 + 4x - 32 \)
   \( y^2 - 10y - 24 \)
   \( a^2 + 2a - 3 \)
   \( x^2 + 9x + 18 \)

3. Factorize the following expressions.
   \( y^2 + 2y + 1 \)
   \( x^2 - 10x + 25 \)
   \( x^2 - 9 \)
   \( x^2 - 49 \)

4. Factorize the following expressions.
   \( x^2 + 16xy + 64y^2 \)
   \( 4x^2 + 4xy + y^2 \)
   \( 9a^2 - 6a + 1 \)
   \( 4x^2 - 81y^2 \)

5. Factorize the following expressions.
   \( 4x^2 - 16y^2 \)
   \( 2x^2 - 12x + 18 \)
   \( 3x^2 - 3x - 6 \)
   \( x^2y - 8xy + 15y \)

Application

6. Use your ingenuity to calculate the following.
   \( 58 \times 62 \)
   \( 96^2 \)

7. If \( x = \sqrt{2} - 3 \), find the value of \( x^2 + 6x + 8 \).
Chapter Summary Problems A

1. Simplify the following.
   1. \(2a(a - 2b)\)
   2. \((6x^2 - 3x) \div (-3x)\)
   3. \((3ab - 9b^2) : \frac{3}{4}b\)
   4. \(5x(x - 1) - x(4x + 5)\)

2. Expand the following expressions.
   1. \((x + 4)(x + 5)\)
   2. \((a + 8)(a - 4)\)
   3. \((3a + 1)^2\)
   4. \((2x + 7)(2x - 7)\)
   5. \((a - 9b)(2a - 7b)\)
   6. \((-4a - b)^2\)

3. Simplify the following.
   1. \((a - 3)^2 - (a + 4)(a - 4)\)
   2. \((x + 7)^2 - (x - 6)(x - 2)\)

4. Simplify the following.
   1. \((2\sqrt{5} - \sqrt{3})^2\)
   2. \((\sqrt{7} + \sqrt{2})(\sqrt{7} - 3\sqrt{2})\)

5. Factorize the following expressions.
   1. \(4m^2n + 2mn\)
   2. \(x^2 + 4x - 5\)
   3. \(x^2 - 11x + 24\)
   4. \(x^2 - 18x + 81\)
   5. \(x^2 + 12x + 36\)
   6. \(64a^2\)
   7. \(3x^2 - 12x - 36\)
   8. \(xy^2 - 9x\)
   9. \(25x^2 - 9y^2\)
   10. \(x^2 - 8xy + 16y^2\)

6. Given two consecutive integers, if we subtract the square of the lesser number from the square of the greater number, the difference is the sum of the original two integers.
   Prove that this result is true.
Chapter Summary Problems

B

1. If \( x = 1 + \sqrt{5} \), \( y = 1 - \sqrt{5} \), find the values of the following expressions.
   \[ \begin{align*}
   (1) & \quad xy \\
   (2) & \quad x^2 - y^2 \\
   (3) & \quad x^2 - 2x + 3
   \end{align*} \]

2. Two natural numbers whose tens digits are both \( a \) and whose unit digits are \( b \) and \( c \) are:
   \[ 10a + b, \quad 10a + c \]

   If \( b + c = 10 \), we can find the product of the two numbers as follows.
   \[ \begin{align*}
   1 & \quad \text{Calculate } a(a + 1), \text{ and write this with the last digit in the hundreds position.} \\
   2 & \quad \text{Write the product of } b \text{ and } c \text{ in the tens and units positions.}
   \end{align*} \]

   Now answer the following questions.
   \[ \begin{align*}
   1 & \quad \text{Calculate } 58 \times 52 \text{ and } 15^2 \text{ by the above method, and check that the answers are correct.} \\
   2 & \quad \text{Prove that the calculation by steps 1 and 2 is correct.}
   \end{align*} \]

Let's investigate!

In diagrams 1 and 2 below, if \( S \) is the area and \( a \) is the width of the white portion, and \( \ell \) the length of the line down the middle, then as in Example 4 and Problem 6 on page 50, \( S = a\ell \).

For a shape of constant width, the area is the product of the width and the length of the line down the middle. Check that this holds for the shapes of constant width shown in diagrams 1 and 2.
1 Using Algebraic Variables

Let's look at the expansion of \((x + y - 2)(x + y + 2)\).

If we replace a part of the expression by a new variable to obtain a simpler expression, we can use this for expansion or factorization.

Example 1

Let's try expanding \((x + y - 2)(x + y + 2)\).

Setting \(x + y = A\), we have:

\[(x + y - 2)(x + y + 2) = (A - 2)(A + 2) = A^2 - 4\]

Now we substitute back \(x + y\) for \(A\) to obtain:

\[(x + y)^2 - 4 = x^2 + 2xy + y^2 - 4\]

Problem 1

Expand the following expressions.

1. \((x + y + 3)(x + y - 5)\)
2. \((a + b + c)^2\)
3. \((a - b - 6)^2\)
4. \((a - b + 4)(a + b - 4)\)

Example 2

Let's try factorizing \((x + y)^2 + 3(x + y) + 2\).

Setting \(x + y = A\), we have:

\[(x + y)^2 + 3(x + y) + 2 = A^2 + 3A + 2 = (A + 1)(A + 2) = (x + y + 1)(x + y + 2)\]

Problem 2

Factorize the following expressions.

1. \((a + b)^2 + 5(a + b) + 6\)
2. \((a - 5)^2 - (a - 5) - 12\)
3. \((2x + 7)^2 - (x - 3)^2\)
4. \(ax - ay - bx + by\)
Making Rectangles

Solve the following puzzle, following the rules carefully.

**Rules**

- Divide up the square on the right along the broken lines, into rectangles (which may be squares) such that each rectangle includes just one of the circled numerals.
- Make the numeral included in each rectangle correspond to the number of small squares in the rectangle.

Rearrange the rectangles below to make a single rectangle.

You can cut out the version printed on p 209.
本書は、東京書籍株式会社発行の検定済教科書『新編 新しい数学 3』（代表著作者：杉山 吉茂・倉野 博／平成17年2月3日検定済）を、同社の承諾を得て、学校法人太田国際学園が英訳し、発行したものです。

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MATHEMATICS

3 Quadratic Equations
4 The Function $y=ax^2$
In a square plot of land, two paths of width 1 meter are arranged as shown in the diagram on the right, leaving the remaining area as flower beds. The resulting area of the flower beds is 35 square meters. Find the length of a side of the square plot of land.

We don't know the widths of three flower bed strips...

Perhaps we can find them somehow?

Tulip fields (Toyama Prefecture)
Let's imagine we moved the paths as shown in the diagram below. If one side of the square plot of land is $x$ meters, answer the following questions.

1. Express the width and length of the flower bed area using $x$.
2. From 1), what equation can we derive?
3. By substituting various values of $x$ into the equation obtained in 2), find the value of $x$ which satisfies the equation.

<table>
<thead>
<tr>
<th>$x$</th>
<th>LHS</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Quadratic Equations

The equation derived in the example on the previous page is as follows, with the terms collected together:

\[ x^2 - 2x - 35 = 0 \]  

The LHS (left-hand side) of (1) is a quadratic expression (that is, an expression of degree 2).

We call an equation a **quadratic equation** if it can be arranged in this way:

\[(\text{quadratic expression}) = 0\]

Rearrange the equation so that the RHS is zero.

**Problem**

Which of the following equations are quadratic equations?

1. \[ x^2 + 4x - 4 = 0 \]
2. \[ (x + 5)(x - 2) = x^2 \]
3. \[ x^2 - 4x = x^2 + 5 \]
4. \[ x^2 - 6 = 0 \]

---

**Let's investigate values of \( x \) that satisfy a quadratic equation.**

As you found on the previous page, the quadratic equation \( x^2 - 2x - 35 = 0 \) holds for the value \( x = 7 \).

**Problem 2**

Say which of -4, -5, -6 also satisfies equation (1).

A value of the variable that satisfies a quadratic equation is called a **solution** (or **root**) of the equation, and the process of finding all solutions is called **solving the equation**.

\( x = 7 \) and \( x = -5 \) are both roots of the quadratic equation (1).

**Check 1**

Say which values from -2, -1, 0, 1, and 2 are solutions of the quadratic equation \( x^2 - x - 2 = 0 \).
How to Solve Quadratic Equations

We'll look at different forms of quadratic equation, and find the best method of solving each one.

Solving by Factorization

For the equation \( x^2 - 2x - 35 = 0 \) on the previous page, factoring the LHS gives \((x - 7)(x + 5) = 0\). In this way, if we can factor the LHS of a quadratic equation, we can use the following fact to solve it.

**For two numbers \( A \) and \( B \),

If \( AB = 0 \) then either \( A = 0 \) or \( B = 0 \).

Example 1

Let's solve \((x - 7)(x + 5) = 0\).

Since this equation expresses the fact that the product of \( x - 7 \) and \( x + 5 \) is zero, one of these two factors must be zero.

In other words:

\[ x - 7 = 0 \quad \text{or} \quad x + 5 = 0 \]

And in either case the equation \((x - 7)(x + 5) = 0\) holds. Therefore the solutions are

\[ x = 7 \quad \text{and} \quad x = -5 \]

Check 1

Solve the following equations.

\[ 1 \quad (x - 2)(x + 4) = 0 \quad \text{and} \quad 2 \quad (x + 5)(x + 3) = 0 \]

Problem 1

Solve the following equations.

\[ 1 \quad x(x - 5) = 0 \quad \text{and} \quad 2 \quad (x + 1)(2x - 1) = 0 \]
Solve \( x^2 - 6x + 5 = 0 \).

Answer

Factoring LHS gives:

\[(x-1)(x-5) = 0\]

\[x-1 = 0 \quad \text{or} \quad x-5 = 0\]

\[x = 1, \ x = 5\]

Check 2 Solve the following equations.

1. \( x^2 + 6x + 8 = 0 \)
2. \( x^2 - 5x + 6 = 0 \)

Problem 2 Solve the following equations.

1. \( x^2 - 4x - 21 = 0 \)
2. \( x^2 - x - 56 = 0 \)
3. \( x^2 - 6x = 0 \)
4. \( x^2 - 13x + 36 = 0 \)
5. \( x^2 + 7x + 12 = 0 \)
6. \( x^2 + 7x - 18 = 0 \)

Example 3 Solve \( x^2 + 6x + 9 = 0 \).

Answer

Factoring the LHS gives:

\[(x+3)^2 = 0\]

\[x+3 = 0\]

\[x = -3\]

Answer \( x = -3 \)

Note Some quadratic equations like the above example have only one solution.

Check 3 Solve \( x^2 + 4x + 4 = 0 \).

Problem 3 Solve the following equations.

1. \( x^2 - 10x + 25 = 0 \)
2. \( x^2 + 14x + 49 = 0 \)
Let's look at how to solve quadratic equations in different forms.

**Example 4** Solve \((x - 4)(x + 1) = -6\).

**Hint** Expand the LHS, then convert to the form \((\text{quadratic expression}) = 0\) before solving.

**Answer**

\[(x - 4)(x + 1) = -6\]

Expand LHS

\[x^2 - 3x - 4 = -6\]

Moving the -6 term and simplifying

\[x^2 - 3x + 2 = 0\]

Factor LHS

\[(x - 1)(x - 2) = 0\]

\[x - 1 = 0\] or \[x - 2 = 0\]

\[x = 1, x = 2\]

**Problem 4** Solve the following equations.

1. \[x^2 = 2(x + 12)\]
2. \[(x - 2)(x - 3) = 20\]
3. \[(x + 8)(x + 2) = 2x\]
4. \[(x + 2)^2 = x + 4\]

**Example 5** Solve \(2x^2 + 10x + 8 = 0\).

**Hint** Divide both sides by the coefficient of \(x^2\), to obtain an equation where the coefficient of \(x^2\) is 1.

**Problem 5** Solve the equation of Example 5.

**Problem 6** Solve \(-3x^2 - 12x + 15 = 0\).

**Let's try!**

To solve the quadratic equation \(x^2 = 3x\), I divided both sides by \(x\), to obtain the solution \(x = 3\). Is this method of solving OK?
Let's find the square root of 5. Then let's find the square root of 9.

We have learnt square roots on page 6.

Let's solve a quadratic equation by considering square roots.

Example 6
Solve \( x^2 - 7 = 0 \).

Moving the \(-7\) term:
\[
x^2 = 7
\]

This shows that \( x \) is a square root of 7, so:
\[
x = \pm \sqrt{7}
\]

Note
We write \( x = \pm \sqrt{7} \) to indicate both of \( x = \sqrt{7} \) and \( x = -\sqrt{7} \) together.

Example 7
Solve \( 4x^2 = 3 \).

Note
\[
4x^2 = 3
\]
\[
x^2 = \frac{3}{4}
\]
\[
x = \pm \sqrt{\frac{3}{4}} = \pm \frac{\sqrt{3}}{2}
\]

Answer \( x = \pm \frac{\sqrt{3}}{2} \)

Check
Solve the following equations.

1. \( x^2 - 2 = 0 \)
2. \( 9x^2 = 5 \)

Problem 7
Solve the following equations.

1. \( x^2 - 36 = 0 \)
2. \( 2x^2 = 8 \)
3. \( 5x^2 = 15 \)
4. \( 25x^2 - 7 = 0 \)
In the trigger question on page 56, if we move the two paths as shown in the diagram on the right, then the shaded portion is a square. Let's write the length of a side of this square expressed in terms of $x$. In this case, what is the area of the square, in square meters?

Given a quadratic equation in the form $(x + \Delta)^2 = \bullet$, we can regard the content of the parentheses as a single entity, and solve using the method on the previous page.

**Example 8** Let's solve $(x - 1)^2 = 36$.

Since $x - 1$ is a square root of 36, we have

$\begin{align*}
(x - 1) &= 6, \\
(x - 1) &= -6
\end{align*}$

And therefore

$\begin{align*}
x &= 7, \\
x &= -5
\end{align*}$

**Example 9** Solve $(x - 3)^2 - 5 = 0$.

**Answer**

$(x - 3)^2 - 5 = 0$

$(x - 3)^2 = 5$

$x - 3 = \pm \sqrt{5}$

$x = 3 \pm \sqrt{5}$

**Answer** $x = 3 \pm \sqrt{5}$

**Note**

We write $x = 3 \pm \sqrt{5}$ to indicate both of $x = 3 + \sqrt{5}$ and $x = 3 - \sqrt{5}$ together.

**Check 5** Solve the following equations.

1. $(x - 2)^2 = 25$
2. $(x - 7)^2 - 6 = 0$

**Problem 8** Solve the following equations.

1. $(x + 5)^2 - 4 = 0$
2. $(x + 6)^2 = 18$
Solving by Converting to the Form \((x + \Delta)^2 = \) 

For a quadratic equation \(x^2 + px + q = 0\), even if we cannot factorize the left-hand side, by converting to the form \((x + \Delta)^2 = \) we can solve it by considering a square root.

Let's first consider the equation above when the coefficient of \(x\) is an even number.

**Example 10** Let's convert \(x^2 + 6x - 5 = 0\) to the form \((x + \Delta)^2 = \).

Moving -5 to the RHS
\[x^2 + 6x = 5\]

Now to convert the LHS of \(x^2 + 6x = 5\) to the form \((x + \Delta)^2\), we add the square of half of 6, the coefficient of \(x\). Thus, by adding 3² to both sides we obtain:
\[x^2 + 6x + 3^2 = 5 + 3^2\]
\[(x + 3)^2 = 14\]

This method is called completing the square.

**Problem 9** In Example 10, solve \((x + 3)^2 = 14\) to find the solutions to \(x^2 + 6x - 5 = 0\). Check that the values you have obtained are the solutions by substituting them for \(x^2 + 6x - 5 = 0\).
Quadratic Equations with an Odd Coefficient of $x$

In the quadratic equation $x^2 + px + q = 0$, even when the coefficient of $x$ is odd we can still solve the equation by the same method as when the coefficient is even.

Let's solve $x^2 + 5x - 3 = 0$.

- Moving $-3$ to the RHS
  
  $$x^2 + 5x = 3$$

Now add the square of half of the coefficient of $x$ to both sides to obtain:

$$x^2 + 5x + \left(\frac{5}{2}\right)^2 = 3 + \left(\frac{5}{2}\right)^2$$

$$\left(x + \frac{5}{2}\right)^2 = \frac{37}{4}$$

$$x + \frac{5}{2} = \pm\sqrt{\frac{37}{4}}$$

$$x + \frac{5}{2} = \pm\frac{\sqrt{37}}{2}$$

$$x = -\frac{5 \pm \sqrt{37}}{2}$$

**Problem 1** Find the numbers to fit in the blue shaded boxes below.

1. $x^2 + 10x + \quad = (x + \quad)^2$
2. $x^2 - 3x + \quad = (x - \quad)^2$

**Problem 2** Following Example 10 on page 64 and the above explanation, solve the following quadratic equations.

1. $x^2 + 4x - 4 = 0$
2. $x^2 - 5x + 2 = 0$
Basic Exercises

Solutions of quadratic equations

Which of the following equations have 3 as a solution?

1. \((x + 3)(x - 5) = 0\)
2. \(x^2 = 3\)
3. \(x^2 + 3x - 18 = 0\)
4. \((x - 9)^2 = 0\)

Solving by factorization

Solve the following equations.

1. \((x + 7)(x - 2) = 0\)
2. \(x^2 + 2x - 24 = 0\)
3. \(x^2 + 3x + 2 = 0\)
4. \(x^2 + 8x = 0\)
5. \(x^2 - 3x - 4 = 0\)
6. \(x^2 - 9x + 18 = 0\)
7. \(x^2 - 4x + 4 = 0\)
8. \(x^2 + 12x + 36 = 0\)

Solving by considering square roots

Solve the following equations.

1. \(x^2 - 5 = 0\)
2. \(7x^2 = 28\)
3. \((x - 5)^2 = 9\)
4. \((x - 1)^2 - 7 = 0\)

Window on math — Seki Takakazu and quadratic equations

Problems involving quadratic equations also appeared in the independent Japanese mathematics (wasan) that flourished in the Edo period. One famous mathematician of this era was Seki Takakazu, who regarded quadratic equations having two solutions or no solutions at all as pathological, rearranging the problems to have a single solution.

Postage stamp featuring Seki Takakazu
Using Quadratic Equations

In the calendar shown on the right, find where two vertically adjacent numbers have a product of 60.

Using Quadratic Equations

Given two different numbers, whose difference is 7 and whose product is 144. Find these two numbers.

Answer: Let the smaller number be \( x \), then the larger number is \( x + 7 \). Since the product of the two numbers is 144, we have:

\[ x(x + 7) = 144 \]

Solving this gives:

\[ x^2 + 7x - 144 = 0 \]

\[ (x - 9)(x + 16) = 0 \]

\[ x = 9, \quad x = -16 \]

When \( x = 9 \), the larger number is \( 9 + 7 = 16 \).

When \( x = -16 \), the larger number is \( -16 + 7 = -9 \).

Answer: 9 and 16 or -16 and -9

Problem 1: Check that both solutions found above fit the conditions of Example 1.

Problem 2: In Example 1, create the equation with \( x \) representing the larger number, and solve it.

Check: Instead of adding 3 to \( x \) then squaring, I added 3 to \( x \) then multiplied by 2. But the result was the same. Find the value of \( x \).
The length of a rectangular piece of paper is 4 cm longer than its width. From each corner of this piece of paper, a square of side 3 cm is cut, and the resulting shape is formed into a cuboidal container, whose volume is found to be 96 cm³. Find the length of the sheet of paper.

**Hint**

Let the width of the sheet of paper be \( x \) cm, then find the length, width, and height of the cuboidal container. Then form an equation for the volume.

**Answer**

If the length of the sheet of paper is \( x \) cm, we have:

\[
3(x-6)(x-2) = 96
\]

\[
x^2 - 8x - 20 = 0
\]

\[
(x-10)(x+2) = 0
\]

\[
x = 10, \ x = -2
\]

In this case, since we must have \( x > 6 \), the answer is 10 cm.

**Problem 3** In Example 2, explain why we must have \( x > 6 \).

When solving applied problems using equations, as in this example, the solutions of the equation may not all be answers to the original problem.

**Problem 4** A rope is stretched around the periphery of a rectangular flower bed of area 30 m², and the length of the rope is found to be 26 m. Find the width of the flower bed.
Let's look at problems involving a point moving along the edge of a geometrical shape.

**Example 3**

As shown in the diagram on the right, a point P starts from point A in a square ABCD, moving along AB toward B. A point Q starts from point D at the same time, and moves along DA toward A at the same speed as point P.

How far will point P have moved from A when the area of \( \triangle APQ \) is 4 cm\(^2\)?

**Hint**

If \( AP = x \) cm then the length of \( AQ \) is \((6 - x)\) cm.

**Answer**

Let \( AP = x \) cm, then since the area of \( \triangle APQ \) is 4 cm\(^2\), we have:

\[
\frac{1}{2}x(6-x) = 4
\]

Multiplying both sides by 2 and expanding:

\[
x^2 - 6x + 8 = 0
\]

\[
(x-2)(x-4) = 0
\]

\[
x = 2, \quad x = 4
\]

Answers 2 cm and 4 cm

**Problem 5**

In Example 3, when the area of \( \triangle APQ \) is 4.5 cm\(^2\), how many centimeters has point P moved from A?

At this point, what kind of triangle is \( \triangle APQ \)?

**Problem 6**

In the diagram on the right, a point P starts from point A in a right-angled isosceles triangle ABC, moving along AB toward B. A point Q starts from point C at the same time point P starts from A, and moves along BC toward B at the same speed as point P.

How many centimeters has point P moved from point A when the area of the trapezoid APQC is 14 cm\(^2\)?
Using quadratic equations (1)

1

The sum of a number and its square is 90. Find the number.

Using quadratic equations (2)

2

As shown in the diagram on the right, the height of a square was decreased by 3 cm and the width was increased by 4 cm, to form a rectangle of area 60 cm². Find the length of a side of the original square.

Window on math — Babylonian clay tablet problems

The following problem was written on a clay tablet around 3800 years ago.

The sum of the length and width of a rectangle is 20. Its area is 96. What is the length and width of the rectangle?

A Babylonian answered this by the following procedure.

1. Take half of the sum of the length and width.
2. Take the square of the value in 1.
3. Subtract the area from the value in 2.
4. Take the positive square root of the value in 3.
5. Add the value in 1 to the value in 4 to obtain the width.
6. Subtract the value in 1 from the value in 4 to obtain the length.

1. Write an equation with \(x\) as the length, and find the length and width.
2. Find the length and width by the above procedure.
1. Solve the following equations.
   
   1. \(2x^2 = 32\)
   2. \((x-2)^2 - 7 = 0\)
   3. \(x^2 - 4x - 12 = 0\)
   4. \(x^2 = 15x\)
   5. \(x^2 + 16x + 63 = 0\)
   6. \(x^2 - 12x + 36 = 0\)
   7. \(x^2 - 12x + 32 = 0\)
   8. \(x^2 - 25 = 0\)
   9. \(x^2 + 6x - 27 = 0\)

2. One root of the quadratic equation \(x^2 + ax - 8 = 0\) is \(-2\)

   1. Find the value of \(a\).
   2. Find the other root.

3. Taking the squares of three successive integers and adding them together gives 302.

   Find the three integers.

4. An \(n\)-gon (polygon with \(n\) sides) has a total of \(\frac{n(n-3)}{2}\) diagonals.

   1. How many diagonals does a hexagon have?
   2. How many sides has a polygon which has 27 diagonals?

5. A rectangular plot of land is 35 meters long and 26 meters wide. As shown in the diagram on the right, paths of the same width are made lengthwise and crosswise through the plot. The remainder is used for growing vegetables.

   How wide should the paths be if the area for growing vegetables is 850 square meters?
**Chapter Summary Problems B**

1. If the roots of the quadratic equation \( x^2 + ax + b = 0 \) are 2 and 3, find the values of \( a \) and \( b \).

2. In the diagram on the right, \( P \) is a point on the graph of \( y = x + 2 \), and \( A \) is the point on the \( x \)-axis such that \( PO = PA \). Taking the \( x \)-coordinate of \( P \) as \( a \), find the following coordinates. Assume that \( a > 0 \), and that the coordinate values are in centimeters.

   1. The \( y \)-coordinate of \( P \).
   2. The coordinates of \( A \).
   3. The coordinates of \( P \) when the area of \( \triangle POA \) is 15 cm\(^2\).

---

*Let's investigate! Let's think!*

You have now studied linear equations in the first year, simultaneous equations in the second year, and quadratic equations in the third year. Look for ways in which the different types of equation are similar and ways in which they are different.

---

*What kind of applied problems do these fit?*
Quadratic Equations of the Form \( x^2 + px + q = 0 \)

Let's look at how to solve a quadratic equation of the form \( x^2 + px + q = 0 \).

\[
x^2 + px + q = 0
\]

Moving \( q \) from the left-hand side to the right-hand side, and adding \( \left( \frac{p}{2} \right)^2 \) to both sides gives:

\[
\begin{align*}
x^2 + px + \left( \frac{p}{2} \right)^2 &= -q + \left( \frac{p}{2} \right)^2 \\
(x + \frac{p}{2})^2 &= -\frac{4q}{4} + \frac{p^2}{4} \\
(x + \frac{p}{2})^2 &= \frac{p^2 - 4q}{4} \\
x + \frac{p}{2} &= \pm \sqrt{\frac{p^2 - 4q}{2}} \\
x &= -\frac{p}{2} \pm \sqrt{\frac{p^2 - 4q}{2}}
\end{align*}
\]

The root of the quadratic equation \( x^2 + px + q = 0 \) is:

\[
x = -\frac{p}{2} \pm \sqrt{\frac{p^2 - 4q}{2}} \quad \text{.................................................. (1)}
\]

Using formula (1), let's find the roots of a quadratic equation. It helps to calculate \( p^2 - 4q \) first.

**Example 1** Let's solve \( x^2 + 3x - 5 = 0 \)

Since \( p = 3 \) and \( q = -5 \),

\[
p^2 - 4q = 3^2 - 4 \times (-5) = 29
\]

And therefore:

\[
x = -\frac{3 \pm \sqrt{29}}{2}
\]
Let's solve $x^2 - 6x + 2 = 0$

Since $p = -6$ and $q = 2$

$$p^2 - 4q = (-6)^2 - 4 \times 2 = 28$$

And therefore:

$$x = \frac{-(-6) \pm \sqrt{28}}{2}$$

$$= \frac{6 \pm 2\sqrt{7}}{2}$$

$$= 3 \pm \sqrt{7}$$

When the coefficient of $x$ is even, we can reduce the fraction to its lowest terms at the end.

Problem 1

Following Examples 1 and 2, solve the following quadratic equations.

1) $x^2 + 5x - 2 = 0$

2) $x^2 - 8x + 4 = 0$

Let's solve $x^2 + x - 12 = 0$

Since $p = 1$ and $q = -12$,

$$p^2 - 4q = 1^2 - 4 \times (-12) = 49$$

And therefore:

$$x = \frac{-1 \pm \sqrt{49}}{2}$$

$$= \frac{-1 \pm 7}{2}$$

$$= 3$$

$$= -4$$

The roots are $x = 3$ and $x = -4$

As in Example 3, when $p^2 - 4q$ is the square of an integer, the roots are numbers not including root signs. In this case it is also possible to solve by factorization.

Problem 2

Following Example 3, solve the following quadratic equations.

1) $x^2 + 4x - 21 = 0$

2) $x^2 - 8x + 16 = 0$

In Problem 2, $2$, $p^2 - 4q = 0$. In this case there is only one root.
When the coefficient of \( x^2 \) is not 1, by dividing both sides of the equation by the coefficient of \( x^2 \) we convert it to a form in which the coefficient of \( x^2 \) is 1. We can then use formula (1) on page 73.

Let's solve \( 3x^2 + 5x + 1 = 0 \)

Dividing both sides by 3, we get:

\[
x^2 + \frac{5}{3}x + \frac{1}{3} = 0
\]

\[
p^2 - 4q = \left( \frac{5}{3} \right)^2 - 4 \times \frac{1}{3} = \frac{13}{9}
\]

And therefore:

\[
x = \frac{-\frac{5}{3} \pm \sqrt{\frac{13}{9}}}{2} = \frac{-5 \pm \sqrt{13}}{6}
\]

Problem 3

Following the above explanation, solve \( 2x^2 - 3x - 4 = 0 \)

If we solve the quadratic equation \( ax^2 + bx + c = 0 \) using the above method, we obtain the following result. This is the general formula for the roots of a quadratic equation.

\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]

Problem 4

For the quadratic equation \( 3x^2 + 7x + 1 = 0 \), answer the following questions.

1. To use the general formula for the roots, what values should be substituted for \( a \), \( b \), and \( c \)?

2. Solve the equation using the general formula.

Problem 5

Solve the following equations.

1. \( x^2 + 6x + 1 = 0 \)
2. \( 2x^2 + 5x - 3 = 0 \)
When a roller-coaster is descending an incline, how does the distance it has traveled vary over time? Instead of testing a roller-coaster, we will experiment in the classroom by rolling a ball down an inclined plane, and investigate what happens.

The roller-coaster first climbs at a steady speed.

A ball was rolled down an inclined plane, and its position plotted every second, with the results in the diagram below. What is the relation between the elapsed time and the distance rolled?
As the roller-coaster descends, it accelerates.

Taking the distance rolled after \( x \) seconds as \( y \) meters, fill in the blanks in the following table from the diagram on the opposite page.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td>0</td>
<td>0.2</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Then let's consider how many meters the ball will roll in 5 seconds.
The Function \( y = ax^2 \)

The following table contains the values of \( x \) and \( y \) from the question on the previous page. Find the values of \( x^2 \), and look at the relation between \( x^2 \) and \( y \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td>0</td>
<td>0.2</td>
<td>0.8</td>
<td>1.8</td>
<td>3.2</td>
</tr>
</tbody>
</table>

As you can see from this table, when we compare corresponding values of \( x^2 \) and \( y \), the value of \( y \) is 0.2 times the value of \( x^2 \).

Therefore, \( y \) is proportional to \( x^2 \), which we can express with the following equation:

\[
y = 0.2x^2
\]

Generally, when \( y \) is a function of \( x \) expressed as \( y = ax^2 \), we say that \( y \) is proportional to the second power of \( x \).

Example 1

If the volume of a square prism with a base of side \( x \) cm and height 5 cm is \( y \) cm\(^3\), \( y \) can be expressed as follows, and \( y \) is proportional to the second power of \( x \):

\[
y = 5x^2
\]

Check 1

A cube has a side of length \( x \) cm. For the following three cases, express \( y \) in terms of \( x \). In which cases is \( y \) proportional to the second power of \( x \)?

1. The sum of the lengths of all the edges is \( y \) cm.
2. The surface area is \( y \) cm\(^2\).
3. The volume is \( y \) cm\(^3\).
Problem 1

A set square and a writing pad are overlapped, as shown in the diagram on the right.
When the distance AP is \( x \) cm, let the area overlapping be \( y \) cm\(^2\), and answer the following questions.

1. Express \( y \) in terms of \( x \).
2. With the expression you found in (1), fill in the blanks in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Example 2

If \( y \) is proportional to the second power of \( x \) and when \( x = 3 \), \( y = 27 \), express \( y \) in terms of \( x \).

Answer

Since \( y \) is proportional to the second power of \( x \), \( y = ax^2 \)

When \( x = 3 \), \( y = 27 \), and therefore substituting these values we have:

\[ 27 = a \times 3^2 \]

\[ a = 3 \]

Therefore \( y = 3x^2 \)

Answer \( y = 3x^2 \)

Check 2

If \( y \) is proportional to the second power of \( x \) and when \( x = 2 \), \( y = -8 \), express \( y \) in terms of \( x \).

Problem 2

\( y \) is proportional to the second power of \( x \) and when \( x = 2 \), \( y = -2 \).

1. Express \( y \) in terms of \( x \).
2. Find the value of \( y \) when \( x = -4 \).
The Graph of $y = ax^2$

The Graph of $y = x^2$

In the equation $y = x^2$, find the value of $y$ for each value of $x$ to complete the blanks in the following table. Then plot the points with corresponding $x$ and $y$ values in the graph below.

<table>
<thead>
<tr>
<th>$x$</th>
<th>$-3$</th>
<th>$-2$</th>
<th>$-1$</th>
<th>$0$</th>
<th>$1$</th>
<th>$2$</th>
<th>$3$</th>
<th>$...$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$...$</td>
</tr>
</tbody>
</table>

It is not possible to draw an accurate graph with the points in the graph on the left alone. To see the area near the origin more clearly, we investigate in more detail, as on the following page.
Problem 1  Investigate points closer to the origin, as follows.

1. Find the value of \( y \) for each value of \( x \) to complete the blanks in the following table.

\[
\begin{array}{cccccccccc}
 x & -1 & -0.9 & -0.8 & -0.7 & -0.6 & -0.5 & -0.4 & -0.3 & -0.2 & -0.1 \\
 y & 0 & 0.5 & 0.6 & 0.7 & 0.8 & 0.9 & 1
\end{array}
\]

2. Plot the points with corresponding \( x \) and \( y \) values in the lower graph on the previous page.

When we have plotted a large number of points with coordinates \( x \) and \( y \) satisfying \( y = x^2 \), we can draw the graph of \( y = x^2 \) as a smooth line as shown on the following page.

Let's look at properties of the function \( y = x^2 \).

We can see the following facts about the graph of \( y = x^2 \):

1. It is symmetrical about the \( y \)-axis.
2. It passes through the origin, without going below the \( x \)-axis.

Problem 2  From fact 2 above, find the range of \( y \).

As you can see from the graph of \( y = x^2 \), we can say the following about the function \( y = x^2 \).

3. As \( x \) increases
   - When \( x < 0 \), \( y \) is decreasing.
   - When \( x > 0 \), \( y \) is increasing.
   - When \( x = 0 \), \( y \) has its minimum value of 0.
The Function $y = ax^2$
The Graph of \( y = ax^2 \)

Let's draw the graph of \( y = 2x^2 \).

For \( y = 2x^2 \), find the numbers to fill the blanks in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>(-2)</th>
<th>(-1.5)</th>
<th>(-1)</th>
<th>(-0.5)</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

As you see from your investigation of the trigger question, for any value of \( x \), the value of \( y \) is twice the value of \( x^2 \).

Therefore, to draw the graph of \( y = 2x^2 \), we can just take the points on the graph of \( y = x^2 \) and double the \( y \)-coordinate.

Check 1

In the diagram on the opposite page, based on the graph of \( y = x^2 \), draw in the graph of \( y = 2x^2 \).

Problem 3

As in Check 1, in the diagram on the opposite page, draw in the graph of \( y = \frac{1}{2}x^2 \).
Now let's draw the graph of \( y = -2x^2 \).

For \( y = -2x^2 \), find the numbers to fill the blanks in the table below.

<table>
<thead>
<tr>
<th>( x )</th>
<th>-2</th>
<th>-1.5</th>
<th>-1</th>
<th>-0.5</th>
<th>0</th>
<th>0.5</th>
<th>1</th>
<th>1.5</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x^2 )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( y )</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

When we compare the equations

\[ y = 2x^2 \quad \text{and} \quad y = -2x^2 \]

we see that for any value of \( x \), the corresponding values of \( y \) have the same magnitude and opposite signs.

Therefore, the graphs of \( y = 2x^2 \) and \( y = -2x^2 \) are symmetrical about the \( x \)-axis.

**Problem 4** In the diagram on the opposite page, based on the graph of \( y = \frac{1}{2}x^2 \), draw in the graph of \( y = -\frac{1}{2}x^2 \).

**Problem 5** In the diagram on the opposite page, draw in the graph of \( y = \frac{1}{4}x^2 \). Then based on this, draw in the graph of \( y = -\frac{1}{4}x^2 \).

**Problem 6** For \( y = -2x^2 \) and \( y = -\frac{1}{2}x^2 \) what happens to the characteristics listed as 1 to 3 on page 81?
$y = \frac{1}{2} x^2$
Drawing the graphs of $y = ax^2$ with various values for $a$ results in a diagram such as is shown on the right.

Using what we have learned so far, we can summarize the characteristics of the graph of $y = ax^2$ as follows:

<table>
<thead>
<tr>
<th></th>
<th>Graph of $y = ax^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Passes through the origin.</td>
</tr>
<tr>
<td>2</td>
<td>Is symmetrical about the $y$-axis.</td>
</tr>
</tbody>
</table>
| 3 | When $a > 0$ the graph opens upward  
   When $a < 0$ the graph opens downward |
| 4 | The larger the absolute value of $a$ the narrower is the shape of the graph. |
| 5 | The graph of $y = ax^2$ is symmetrical about the $x$-axis with the graph of $y = -ax^2$. |

**Problem 7** Check fact 4 in the above diagram.

*The Function $y = ax^2$*
The graph of $y = ax^2$ is called a **parabola**. It has an axis of symmetry, and the point of intersection of the axis and the parabola is called the vertex of the parabola.

You can see a parabola in the curve traced by a fountain or by fireworks. And there are more examples of a parabola that you will see in everyday life.

---

**Window on math — Parabolic antennas**

A parabolic antenna, used for example to receive satellite broadcasting signals, has a curved surface obtained by rotating a parabola about its axis. This surface has the property that parallel electromagnetic rays impinging on the surface are all focused to a single point.
Ranges of $x$ and $y$

Let's look at the ranges of $x$ and $y$ in the function $y = ax^2$.

**Example 1**

For the function $y = 3x^2$ let's use the graph to look at the range of $y$ when $x$ is in the range $-1 \leq x \leq 2$.

The part of the graph of this function for the range

$$-1 \leq x \leq 2$$

is shown by a thicker line in the diagram on the right. Thus we can see that:

- when $x = 0$, $y$ has the minimum value, 0
- when $x = 2$, $y$ has the maximum value, 12

Therefore the range is

$$0 \leq y \leq 12$$

**Check 2**

For the function $y = 3x^2$, when $x$ is in the range $-2 \leq x \leq 1$, find the range of $y$.

**Problem 8**

For the function $y = 2x^2$, Arthur used the following method to find the range of $y$ when $x$ is in the range $-1 \leq x \leq 3$. Explain where Arthur's reasoning is wrong.

When $x = -1$, $y = 2$, and when $x = 3$, $y = 18$. Therefore the range of $y$ is:

$$2 \leq y \leq 18$$

**Problem 9**

For the function $y = -2x^2$, find the range of $y$ for the following ranges of $x$.

1. $2 \leq x \leq 4$
2. $-2 \leq x \leq 1$

4. The Function $y = ax^1$
Rate of Change

When a roller-coaster is descending an incline, its speed increases progressively. Let’s investigate how the speed varies, by rolling a ball down an inclined plane.

We will consider a slope where the distance $y$ meters traveled by a ball $x$ seconds after it starts rolling satisfies the following equation.

$$y = x^2$$

Problem 1: In the above equation $y = x^2$, find the values of $y$ when $x$ is 3, 4, and 5, and enter them in the following table. Find the distances the ball moves in each second, and write these in the blue boxes.

<table>
<thead>
<tr>
<th>$x$ (s)</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$ (m)</td>
<td>0</td>
<td>1</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Distance traveled in each second (m)

1 — The Function $y = ax^2$
We can show the distance traveled in each second graphically, as in the diagram on the right.

We can find the average speed from the following expression:

\[
\text{average speed} = \frac{\text{distance traveled}}{\text{time elapsed}} \quad \text{(m/s)}
\]

If we find the average speed for each second we get the following:

- From 0 to 1 second elapsed: 1 m/s
- From 1 to 2 seconds elapsed: 3 m/s
- From 2 to 3 seconds elapsed: 5 m/s

From this we see that the speed is increasing.

Example 1

In the example above, find the average speed over the interval from 1 to 3 seconds after the ball starts rolling.

Answer

The distance traveled from 1 to 3 seconds after the start is:

\[
3^2 - 1^2 = 9 - 1 = 8 \quad \text{(m)}
\]

Therefore, the average speed for this interval is:

\[
\frac{9 - 1}{3 - 1} = \frac{8}{2} = 4 \quad \text{(m/s)}
\]

Answer 4 m/s

Problem 2

For the ball rolling on an inclined plane discussed above, find the following.

1. Average speed in the first three seconds after the ball starts rolling.
2. Average speed in the interval from 1 to 5 seconds after the ball starts rolling.

For this example of a ball rolling on an inclined plane, the average speed can be found with the following formula:

\[
\text{(change in } y \text{)} \quad \text{(change in } x \text{)}
\]

\[
\frac{y}{x} \quad \text{(1)}
\]

When \( y \) is a function of \( x \), we call the value of (1) the rate of change.

The Function \( y = ax^2 \)
For the function \( y = \frac{1}{2}x^2 \), find the rate of change when \( x \) increases from 2 to 4.

**Answer**

When \( x = 2 \) \( \quad y = \frac{1}{2} \times 2^2 = 2 \)

When \( x = 4 \) \( \quad y = \frac{1}{2} \times 4^2 = 8 \)

Therefore, the rate of change is:

\[
\frac{\text{change in } y}{\text{change in } x} = \frac{8 - 2}{4 - 2} = \frac{6}{2} = 3
\]

**Check**

For the function \( y = 2x^2 \), find the rate of change when \( x \) increases as follows.

1. From 2 to 4
2. From -5 to -2

**Problem**

For the function \( y = \frac{1}{2}x^2 \), find the rate of change when \( x \) increases as follows.

1. From 0 to 2
2. From 4 to 6
3. From -6 to -4

Let’s compare the rate of change with the case of linear functions.

For the following two functions, choose a number of starting and ending values for \( x \), and find the corresponding rates of change.

1. Linear function \( y = 2x + 1 \)
2. Function \( y = 2x^2 \)

The rate of change of a linear function is constant, as you have already discovered, for the function \( y = ax^2 \) the rate of change depends on the starting and ending values for \( x \).
As you found in Example 2 on the previous page, for the function \( y = \frac{1}{2} x^2 \), the rate of change is 3 when \( x \) increases from 2 to 4. This value is shown in the diagram on the right as the slope of a straight line passing through the two points A (2, 2) and B (4, 8) on the graph of \( y = \frac{1}{2} x^2 \).

**Problem 4** Find the equation for the straight line AB in the diagram on the right.

---

**Instantaneous Speed**

Consider the ball rolling down an inclined plane which we investigated on page 89. Let's consider smaller and smaller intervals, and find the average speeds over them, as follows.

- Elapsed time from 1 s to 1.1 s
- Elapsed time from 1 s to 1.01 s
- Elapsed time from 1 s to 1.001 s
- Elapsed time from 1 s to 1.0001 s

As we make the interval smaller, as above, the average speed approaches the instantaneous speed when one second has elapsed.

**Problem 1** In the case of a ball rolling on an inclined plane for which the equation \( y = 4x^2 \) holds, what do you think is the instantaneous speed one second after the ball starts rolling?
**Comparison with Linear Functions**

Let's summarize the properties of the function $y = ax^2$ and the function $y = ax + b$.

**Problem 5** Enter the appropriate description in the blue boxes in the following table.

<table>
<thead>
<tr>
<th>Form of the graph</th>
<th>Function $y = ax + b$</th>
<th>Function $y = ax^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Slope</td>
<td>$a$</td>
<td>$a$</td>
</tr>
<tr>
<td>Straight line</td>
<td></td>
<td>Vertex</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>When $a &gt; 0$</th>
<th>Increasing</th>
<th>Increasing</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>Increasing</td>
<td>Increasing</td>
</tr>
<tr>
<td>$x$</td>
<td>Increasing</td>
<td>Increasing</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td>Increasing</td>
<td>Increasing</td>
</tr>
<tr>
<td>$a &lt; 0$</td>
<td>Decreasing</td>
<td>Decreasing</td>
</tr>
<tr>
<td>$x$</td>
<td>Decreasing</td>
<td>Decreasing</td>
</tr>
<tr>
<td>$a &lt; 0$</td>
<td>Not constant</td>
<td></td>
</tr>
<tr>
<td>$x = 0$</td>
<td>Changes from decreasing to increasing when $x = 0$.</td>
<td>Changes from decreasing to increasing when $x = 0$.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Change in the value of $y$</th>
<th>When $a &gt; 0$</th>
<th>When $a &lt; 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$y$</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
<tr>
<td>$x$</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
<tr>
<td>$a &gt; 0$</td>
<td>Increasing</td>
<td>Decreasing</td>
</tr>
<tr>
<td>$a &lt; 0$</td>
<td>Not constant</td>
<td></td>
</tr>
<tr>
<td>$x = 0$</td>
<td>Changes</td>
<td>Increases</td>
</tr>
</tbody>
</table>

Constant and equal to $a$ | Not constant

1 — The Function $y = ax^2$ 93
Let's look at everyday situations where the equation $y = ax^2$ occurs.

**Example 1**
If an object is dropped after $x$ seconds, the distance fallen, $y$ meters, is given approximately by the following relation.

$$y = 4.9x^2$$

**Problem 1**
For Example 1, answer the following questions.

1. Approximately how many meters has the object fallen 2 seconds after dropping?
2. If an object is dropped from a height of 80 m, approximately how many seconds will it take to reach the ground?

**Problem 2**
If a pendulum takes $x$ seconds to make one complete vibration has a length of $y$ meters, then the following relation holds:

$$y = \frac{1}{4}x^2$$

Find the length of a pendulum for which one vibration takes one second.

**Example 2**
When the brakes are applied on a vehicle, the distance traveled from the instant the brakes are applied until the vehicle stops is called the braking distance. The braking distance is approximately proportional to the square of the speed of the vehicle.

**Problem 3**
Suppose that when a vehicle is traveling at 40 km/h the braking distance is 12 meters.

1. When the speed is 80 km/h, what is the braking distance?
2. If the braking distance is $y$ meters when the speed is $x$ km/h, express $y$ in terms of $x$.
3. Suppose the braking distance should not be more than 75 meters. In this case, what is the maximum permissible vehicle speed?
Using Graphs

We can use straight-line and parabola graphs on problems such as the following.

Alex and Boris start together from the top of a slope of length 40 meters. Alex descends at a speed of 4 m/s, while Boris freewheels on a skateboard. When $x$ seconds have elapsed from starting to descend, Boris has traveled $\frac{1}{2}x^2$ m. How many seconds does it take for Alex to be passed by Boris?

For Boris to catch up with Alex, the following must be true:
(Distance traveled by Alex) = (Distance traveled by Boris)

If we let $x$ seconds be the elapsed time, we can solve this problem with an equation.

If the distance traveled when $x$ seconds have elapsed from starting is $v$ meters, then we can express the distances traveled by Alex and Boris as follows:

Alex: $v = 4x$

Boris: $v = \frac{1}{2}x^2$

If we read the x-coordinate where these two graphs cross, we can find how many seconds have elapsed when Boris catches up with Alex.

We can represent the distance traveled by Boris by a parabola graph, as in the diagram on the right.
Problem 4  For the problem on the previous page, answer the following questions.

1. Draw a graph representing the movement of Alex in the diagram on the previous page.

2. After how many seconds will Boris catch up with Alex?

Let's look at another problem with a straight line and a parabola.

Problem 5  Three points A, B, and C lie on the graph of $y = x^2$, and their $x$-coordinates are $-2, -1$ and $2$, respectively.

1. Find the equation of the straight line $BC$.

2. Find the equation of the straight line $l$ passing through point A and parallel to BC.

3. Let D be the point other than A at which line $l$ intersects the graph of $y = x^2$. Find the coordinates of D using the graph.

---

**Window on math — Galileo and falling objects**

In ancient times it was widely believed that a heavy object would fall faster than a lighter one. It was the Italian scientist Galileo (1564 - 1642) who showed by experiment that this is not true.

It was also Galileo who discovered from many experiments of rolling a ball down an inclined plane that the distance traveled by a falling object is proportional to the square of the elapsed time.
Basic Exercises

1. **Quantities proportional to a second power**
   - A cylinder with a base of radius \( x \) cm and height 3 cm has a volume of \( y \) cm\(^3\). In this case, write an expression for \( y \) in terms of \( x \).

2. **The function \( y = ax^2 \)**
   - Suppose that \( y \) is proportional to the second power of \( x \), and when \( x = 3 \), \( y = 18 \). In this case, write an expression for \( y \) in terms of \( x \).

3. **The graph of \( y = ax^2 \)**
   - The curves marked 1, 2, and 3 in the diagram on the right represent the graphs of the function 2 to 5 below. Which of the curves 1 to 3 corresponds to which function below.
   - a. \( y = 2x^2 \)
   - b. \( y = -x^2 \)
   - c. \( y = \frac{1}{2}x^2 \)

4. **Ranges of the function \( y = ax^2 \)**
   - For the function \( y = 4x^2 \), find the range of \( y \) for the following two cases of the range of \( x \).
     - (1) \( 1 \leq x \leq 3 \)
     - (2) \( -2 \leq x \leq 1 \)

5. **Rate of change of the function \( y = ax^2 \)**
   - For each of the following two functions, find the rate of change as \( x \) increases from 1 to 3.
     - (1) \( y = 3x^2 \)
     - (2) \( y = -3x^2 \)
Chapter Summary Problems • A

1. For the function \( y = ax^2 \), in each of the following cases find the value of \( a \).
   
   1. When \( x = 4, \ y = 2 \).
   2. The graph is the parabola shown in the diagram on the right.
   3. If the range of \( x \) is \(-1 \leq x \leq 3\), the range of \( y \) is \( 0 \leq y \leq 6 \).
   4. When \( x \) increases from 1 to 2, the rate of change is 6.

\[ \text{Diagram of a parabola} \]

2. For the following functions, find the minimum and maximum values of \( y \) when the range of \( x \) is \(-2 \leq x \leq 3\).

   1. \( y = -3x + 5 \)
   2. \( y = 2x^2 \)

3. For the two functions \( y = 5x - 1 \) and \( y = x^2 \), when \( x \) increases from \( a \) to \( a+3 \), the rate of change of the two functions is the same. Find the value of \( a \).

4. The right triangle ABC in the diagram on the right has point P moving from point B along the side AB to point A. Point Q leaves B at the same time as point P, and moves along side BC to C, at twice the speed of point P.

   If the area of \( \triangle PBQ \) is \( y \) cm\(^2\) when the length of BP is \( x \) cm, answer the following questions.

   1. Write an expression for \( y \) in terms of \( x \).
   2. When \( x = 3 \), find the value of \( y \).
   3. Find the ranges of \( x \) and \( y \).
1. Let $S$ be the area of $\triangle POA$ with vertices at point $P(\pi, y)$ on the graph of $y = x^2$, the origin $O$, and point $A(4, 0)$. Answer the following questions.

   1. Write an expression for $S$ in terms of $x$.
   2. When $S = 50$, find the coordinates of $P$.

2. Let $A$ and $B$ be points on the graph of $y = \frac{1}{4}x^2$ with $x$-coordinates $–4$ and $2$, respectively. Let $C$ be the point where the straight line through $A$ and $B$ intersects the $y$-axis. If $P$ is a point moving along the graph of $y = \frac{1}{4}x^2$, answer the following questions.

   1. Find the area of $\triangle OAB$.
   2. Find the coordinates of $P$ when the area of $\triangle OCP$ is one half the area of $\triangle OAB$.

Let's investigate! Let's think!

In the first year you have studied proportionality and inverse proportionality, in the second year linear functions, and now in the third year the function $y = ax^2$. Compare the properties of these functions.

Answer
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MATHMATICS

3

5 Similar Figures
6 The Pythagorean Theorem
Let's try finding distances on the ground using a map.
In the map on the page on the left, we know that the actual distance between points A and B is 15 km.

1. In the map on the page on the left, distances are reduced to one part in how many? Measure the distance on the map between points A and B, and calculate the ratio by which distances are reduced on the map.

2. Using the scale you calculated in 1, find the actual distance between points B and C.

Using enlarged images we can study small objects, and with reduced images we can see the whole of a large object. Thus we utilize images which either shrink or magnify in our daily lives. Let's look at images on different scales.
How has the shape of quadrilateral (a) below been changed to form the quadrilaterals (b) to (e)?

When a figure is enlarged or shrunk without changing its shape, we say the resulting figure is similar to the original figure. Two shapes are similar if the only difference is size.

In the above diagram, quadrilaterals (f) and (g) are similar figures. Quadrilateral (h) is the mirror image of (g), and thus these two quadrilaterals are congruent. In this case also, we say that (f) and (h) are similar figures.

Comparing quadrilaterals $A'B'C'D'$ and $ABCD$ above, we can see the following relations between corresponding sides and angles.

$A'B' = 2AB, \quad B'C' = 2BC, \quad C'D' = 2CD, \quad D'A' = 2DA$

$\angle A' = \angle A, \quad \angle B' = \angle B, \quad \angle C' = \angle C, \quad \angle D' = \angle D$
We can express the fact that quadrilaterals ABCD and A'B'C'D' are similar figures by writing:

\[ \text{Quadrilateral ABCD} \sim \text{Quadrilateral A'B'C'D'} \]

Here \(\sim\) is the symbol for similarity. When showing the similarity of two polygons, we write the names of corresponding vertices in the same order.

**Check 1** The two triangles below are similar. Express this using the \(\sim\) symbol. Also express symbolically the relations between corresponding edges and angles.

![Triangles](image)

**Problem 1** In the following diagram, draw a quadrilateral EFGH with sides corresponding to those of quadrilateral ABCD enlarged three times. Express symbolically the relations between corresponding edges and angles of quadrilaterals ABCD and EFGH.

![Quadrilateral](image)

For similar figures, the following always holds.

**Properties of similar figures**

For similar figures the ratios of the lengths of the corresponding sides equal, and the magnitudes of the corresponding angles are equal.
Problem 2  
In the following diagram, O is any suitable point, and vertex A' is such that OA' = 2 OA. Find the vertices B', C', and D' similarly, and draw in quadrilateral A'B'C'D'.

Problem 2  
Compare the lengths of corresponding sides and the corresponding angles of quadrilaterals ABCD and A'B'C'D' drawn in the trigger question above, and check that these are similar figures.

When we have two figures like quadrilaterals ABCD and A'B'C'D' drawn as in the trigger question so that straight lines passing through corresponding vertices meet at a point O, and the ratios of the distances from O to corresponding vertices are all equal, then in this diagram we say that these figures are in positions of similarity with respect to O, the center of similarity. Two figures in positions of similarity are similar figures.

Check 2  
In the diagram on the right, the two triangles are in positions of similarity with O as the center of similarity. Express the fact that the triangles are similar, using the \( \sim \) symbol. Say which are the corresponding vertices.

Problem 3  
Draw a quadrilateral ABCD, choose a point O as the center of similarity, and construct another quadrilateral with sides one-half the length of the sides of quadrilateral ABCD.

104 5—Similar Figures
**Ratio of Similarity**

In the diagram on the right, \( \triangle ABC \sim \triangle DEF \). Find the ratio of the lengths of corresponding sides.

For similar figures, the ratio of the lengths of corresponding sides is called the **ratio of similarity**. For example, in the trigger question, the ratio of similarity of \( \triangle ABC \) and \( \triangle DEF \) is 2:1.

**Example 1**

In the diagram on the right, \( \triangle ABC \sim \triangle DEF \), and the ratios of the lengths of corresponding sides are as follows:

\[
\begin{align*}
AB : DE &= 6 : 9 \\
BC : EF &= 8 : 12 \\
CA : FD &= 4 : 6
\end{align*}
\]

And these are all equal to 2:3. The ratio of similarity of \( \triangle ABC \) and \( \triangle DEF \) is 2:3.

**Check 3**

In the diagram on the right,

\( \triangle ABC \sim \triangle DEF \)

Find the ratio of similarity of \( \triangle ABC \) and \( \triangle DEF \).

Two circles are similar figures, and the ratio of similarity is the ratio of their radii.

We can think of congruent figures as similar figures with a ratio of similarity of 1:1.
Properties of Proportion

For the ratio written as \(a:b\), the quotient \(\frac{a}{b}\) is called the value of the ratio.

In Example 1 on the previous page, find the values of the ratios \(AB:DE\), \(BC:EF\), and \(CA:FD\).

For two ratios to be equal means that their values are equal.

Thus \(a:b = m:n\) means that \(\frac{a}{b} = \frac{m}{n}\).

Multiplying both sides of \(\frac{a}{b} = \frac{m}{n}\) by \(bn\), we have:

\[an = bm\]

In other words, the following holds:

**If \(a:b = m:n\) then \(an = bm\)**

**Check 4** Find the value of \(x\) in the following:

\[\begin{aligned}
1) \quad x : 8 &= 3 : 2 \\
2) \quad 9 : 4 &= x : 6
\end{aligned}\]

In the diagram on the right, \(\triangle ABC \sim \triangle A'B'C'\).

The following holds for the ratios of corresponding sides:

\[\frac{6}{3} = \frac{4}{2}\]

Further, the following holds for the ratios of adjacent sides:

\[\frac{6}{4} = \frac{3}{2}\]

The following also always holds:

**If \(a:b = m:n\) then \(a:m = b:n\)**

5 — Similar Figures
Let's find lengths of sides, using the ratio and proportion.

**Example 2**

In the diagram on the right,\[\triangle ABC \sim \triangle DEF\]

Find the length of side DF.

**Hint**

Since the ratios of corresponding sides are equal, then

\[\frac{AB}{DE} = \frac{AC}{DF}\]

**Answer**

If DF = \(x\) cm, then

\[9 : 6 = 12 : x\]

Using proportions:

\[9 \times x = 6 \times 12\]

\[x = 8 \quad \text{Answer} \ 8 \text{ cm}\]

**Note**

You can also solve Example 2 by using \(\frac{AB}{AC} = \frac{DE}{DF}\).

**Check 5**

In the diagram on the right,\[\triangle ABC \sim \triangle DEF\]

Find the length of side AB.

**Problem 4**

In the following diagram, quadrilaterals ABCD and EFGH are similar figures.

1. Find the ratio of similarity of quadrilaterals ABCD and EFGH.
2. Find the lengths of sides AB and EH.
Conditions for Similar Triangles

To show that two triangles are similar, what do we need to know about the relations between their sides and angles?

In the diagram on the right, the distance between points A and B at opposite ends of the pond can be found by choosing a point C from which both A and B can be seen, and by making a map of \( \triangle ABC \). What parts of \( \triangle ABC \) do we need to know to make the map?

In the trigger question above, we measured the actual distances CA and CB, and \( \angle ACB \), and found that CA = 15 m, CB = 18 m, and \( \angle ACB = 78^\circ \).

Based on these measurements, we can draw \( \triangle A'B'C' \) as a 1/300 scale map of \( \triangle ABC \) as follows.

Using the map on the left, find the distance between A and B.

Two sides of a triangle and the included angle determine the triangle. Using this fact, we can draw a similar triangle.
Problem 2  Given △ABC as in the diagram on the right, draw △DEF to meet each of the following two sets of conditions.

1. \[ EF = 2a, \quad FD = 2b, \quad DE = 2c \]
2. \[ EF = 2a, \quad \angle E = \angle B, \quad \angle F = \angle C \]

Each of the triangles DEF you drew in Problem 2 should be similar to △ABC.

From what we have studied so far, we can derive the following conditions for triangles to be similar figures.

<table>
<thead>
<tr>
<th>Conditions for triangles to be similar figures</th>
</tr>
</thead>
<tbody>
<tr>
<td>Two triangles are similar figures if any of the following sets of conditions holds.</td>
</tr>
</tbody>
</table>

1. **Ratios of all three sides are equal.**

   \[ \frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'} \]

2. **Ratios of two sides are equal, and the included angles are equal.**

   \[ \frac{a}{a'} = \frac{c}{c'}, \quad \angle B = \angle B' \]

3. **Two pairs of angles are equal.**

   \[ \angle B = \angle B', \quad \angle C = \angle C' \]
Check 1 From the following figures, select the sets of similar triangles, and say what conditions you used to show similarity.

In the following diagrams, since ΔABC and ΔADB have two pairs of sides in the same ratio and the included angle is equal, ΔABC ∼ ΔADB.

Problem 3 In each of the diagrams below, show which triangles are similar figures, using the ∼ symbol. Also say what conditions you used to show similarity.
Using Similarity

Let's try using the conditions for similar triangles to solve problems.

Example 2
For right triangle ABC, where \( \angle A = 90^\circ \), we drop a perpendicular from A to the side BC. Prove that \( \triangle ABC \sim \triangle DBA \).

Proof
For \( \triangle ABC \) and \( \triangle DBA \):
\[
\begin{align*}
\angle BAC &= \angle BDA = 90^\circ \\
\angle B &= \text{common}
\end{align*}
\]
Since the two pairs of angles are equal,
\( \triangle ABC \sim \triangle DBA \).

Problem 4
In the proof of Example 2, show that \( BC:BA = BA:BD \).

Problem 5
In Example 2, answer the following questions:

1. Prove that \( \triangle ADC \sim \triangle BDA \).
2. From the proof of 1, show that \( AD:CD = BD:AD \).

Problem 6
In \( \triangle ABC \) in the diagram on the right, perpendiculars BD and CE are dropped from vertices B and C to sides AC and AB, respectively.

Now prove that:
\( \triangle ABD \sim \triangle ACE \).
As shown in the diagram on the right, two chords $AB$ and $CD$ are drawn in circle $O$, to intersect at point $P$. In this case, prove that:

$$\triangle ACP \sim \triangle DBP$$

**Hint** Use the fact that inscribed angles intercepting the same arc of a circle are equal.

**Proof**

In $\triangle ACP$ and $\triangle DBP$:

Since vertically opposite angles are equal

$$\angle APC = \angle DPB \quad \cdots \quad (1)$$

Since the inscribed angles intercepted by $CB$ are equal

$$\angle CAP = \angle BDP \quad \cdots \quad (2)$$

From (1) and (2), we have two pairs of equal angles, and thus

$$\triangle ACP \sim \triangle DBP$$

**Problem 7**

As shown in the diagram on the right, chords $AB$ and $CD$ intersect at point $P$. Using the result of Example 3, find the length of $PD$.

**Problem 8**

In the diagram on the right, $A$, $B$, $C$, and $D$ are points on circle $O$, and $AB = AC$. If $P$ is the intersection of chords $AD$ and $BC$, prove that $\triangle ABP \sim \triangle ADB$.
Using properties of similar figures, now we'll find the height of a school building or tree.

**Example 4**

At a point P 16 meters from the school building, looking up at the roof of the building my line of sight makes an angle of 40 degrees with the horizontal. If my eye is 1.5 meters above the ground, find the height of the building.

**Hint**

Draw a scale diagram of \( \triangle ABC \), and from it find the length of AC, then add the height of my eye BP.

**Problem 9**

In Example 4, find the height of the school building.

**Problem 10**

As shown in the diagram on the right, when a stick \( AB \) of height 1 meter casts a shadow \( BC \) of length 60 cm, the tree casts a shadow \( EF \) of length 4.8 meters. Find the height of the tree \( DE \).

**Problem 11**

The photo on the right shows Ayako when she entered elementary school. The height of the wall on the right is 1.6 meters. Find Ayako's height at this time.
Basic Exercises

**Ratio of similarity**

In the diagram on the right, suppose that
\[ \triangle ABC \sim \triangle DEF \]

1. Find the ratio of similarity of \( \triangle ABC \) and \( \triangle DEF \).

2. Find the length of side \( DF \).

**Conditions for similar triangles**

In the following diagram, find the sets of triangles that are similar figures, and express this using the \( \sim \) symbol. Also say what conditions you used to show similarity.

**Using Similarity**

In \( \triangle ABC \) in the diagram on the right, \( D \) is a point on the side \( AB \), and \( \angle A = \angle BCD \).
In this case, prove that \( \triangle ABC \sim \triangle CBD \).
As shown below, we can divide a ruled notebook into three equal columns.

1. Try this out in practice.

2. Try to see why the above method results in three equal columns.

It looks as though we could divide into four or five columns the same way...
The ruled lines on the notebook are parallel and evenly spaced. By thinking of the trigger question on the previous page as in the diagram on the right, we can see similar triangles.

Problem 1 In the diagram above, prove that \( \triangle ABC \sim \triangle ADE \) and that \( \triangle ABC \sim \triangle AFG \). Then find the relationship governing the lengths of BC, DE, and FG.

In \( \triangle ABC \), let a straight line parallel to side BC intersect the two sides AB and BC at points D and E respectively. Then we will investigate the relationship among the sides of \( \triangle ADE \) and \( \triangle ABC \).

Problem 2 In the diagram on the right, suppose DE \( \parallel \) BC.

1. Prove that \( \triangle ADE \sim \triangle ABC \).

2. Looking at corresponding sides of \( \triangle ADE \) and \( \triangle ABC \), say which sides are in the same ratio as AD:AB.

Next let's look at the inverse of the above results.

Problem 3 In the diagram on the right, D and E are points such that \( \frac{AD}{AB} = \frac{AE}{AC} \).

1. Prove that \( \triangle ADE \sim \triangle ABC \).

2. Prove that DE \( \parallel \) BC.
From what we have learned up to now, we can see that the following theorem holds.

### Triangles and proportions (1)

**Theorem**

In \( \triangle ABC \), with points D and E on sides AB and AC, respectively.

1. If \( DE \parallel BC \) then
   \[
   \frac{AD}{AB} = \frac{AE}{AC} = \frac{DE}{BC}
   \]

2. If \( \frac{AD}{AB} = \frac{AE}{AC} \) then
   \( DE \parallel BC \)

Results 1 and 2 in the above theorem also hold if the points D and E are on extensions of sides BA and CA, as shown in the diagram on the right.

#### Example 1

In the diagram on the right, if \( DE \parallel BC \), find the length of \( DE \).

**Answer**

Let \( DE = x \) cm.

Since \( DE \parallel BC \)

\[
\frac{AE}{AC} = \frac{DE}{BC}
\]

4 : 6 = \( x : 9 \)

36 = 6 \( x \)

\( x = 6 \) Answer 6 cm

#### Check 1

In the diagram on the right, if \( DE \parallel BC \), find the length of \( AD \).
In part 1 of the theorem on the previous page, let's investigate the relationship between AD:DB and AE:EC.

Problem 4 In the diagram on the right, suppose DE \parallel BC.

With AD:DB = 3:2, find AD:AB, AE:AC, and AE:EC.

Next, when AD:DB = AE:EC, let's look at the relationship between DE and BC.

Problem 5 In the diagram on the right, if AD:DB = AE:EC = 3:2, answer the following questions.

1. Find each of AD:AB and AE:AC.
2. Prove that DE \parallel BC.

The following theorem always holds.

---

**Triangles and proportions (2)**

**Theorem**

In \( \triangle ABC \), with points D and E on sides AB and AC, respectively.

1. If DE \parallel BC then 
   \[ AD:DB = AE:EC \]

2. If AD:DB = AE:EC then 
   \[ DE \parallel BC \]

---

5—Similar Figures
In the diagram on the right, suppose \( DE \parallel BC \).
Find the length of \( EC \).

**Answer**

Let \( EC = x \) cm.

Since \( DE \parallel BC \)

\[
\frac{AD}{DB} = \frac{AE}{EC}
\]

\[
8 : 12 = 6 : x
\]

\[
8x = 72
\]

\[
x = 9
\]

**Answer 9 cm**

**Check 2** In the diagram on the right, suppose \( DE \parallel BC \).
Find the length of \( AD \).

**Problem 6**
Find the values of \( x \) and \( y \) if \( DE \parallel BC \) in the following diagrams.

**Problem 7**
In the diagram on the right, which of the line segments \( DE \), \( EF \), and \( FD \) is parallel to a side of \( \triangle ABC \)? Explain why.
Triangle Midpoint Theorem

In \( \triangle ABC \), let the midpoints of sides \( AB \) and \( AC \) be \( M \) and \( N \) respectively. In this case, what relations hold between the line segment \( MN \) and the side \( BC \)?

\[ MN \parallel BC \]
\[ MN = \frac{1}{2} BC \]

Problem 8
In \( \triangle ABC \), let the midpoints of sides \( BC \), \( CA \), and \( AB \) be \( D \), \( E \), and \( F \) respectively. Prove that \( \triangle DEF \sim \triangle ABC \). Give all the triangles which are congruent to \( \triangle DEF \).

Problem 9
In the diagram on the right, quadrilateral \( ABCD \) is a trapezoid with \( AD \parallel BC \). Let \( E \) be the midpoint of side \( AB \). Draw a line from \( E \) parallel to side \( BC \) to intersect \( BD \) and \( CD \) at points \( F \) and \( G \) respectively. Find the lengths of \( EF \) and \( EG \).
In quadrilateral ABCD, if the midpoints of sides AB, BC, CD, and DA are E, F, G, and H respectively, then quadrilateral EFGH is a parallelogram. Prove this.

**Hint** Divide quadrilateral ABCD into two triangles by drawing a diagonal, then use the triangle midpoint theorem.

**Proof** Draw the diagonal BD of quadrilateral ABCD. Then in \( \triangle ABD \), E is the midpoint of AB and H is the midpoint of AD. Therefore:

\[
EH \parallel BD, \quad EH = \frac{1}{2} BD
\]

Similarly in \( \triangle CDB \):

\[
FG \parallel BD, \quad FG = \frac{1}{2} BD
\]

And therefore EH \( \parallel FG \) and EH = FG.

Since one pair of sides is parallel and equal, quadrilateral EFGH is a parallelogram.

**Problem 10** As shown in the diagram on the right, in quadrilateral ABCD, AB = CD. Let P be the midpoint of diagonal AC. Let Q and R be the midpoints of sides AD and BC, respectively. In this case, what type of triangle is \( \triangle PQR \)? Prove your answer.
Parallel Lines and Proportions

When straight lines intersect parallel lines, the following theorem holds:

**Theorem**

If three parallel lines \(a, b,\) and \(c\) intersect a straight line \(\ell\) at \(A, B,\) and \(C\) respectively, and intersect a straight line \(\ell'\) at \(A', B',\) and \(C'\) respectively, then

\[
\frac{AB}{BC} = \frac{A'B'}{B'C'}
\]

**Proof**

Draw a straight line through point \(A\) parallel to \(\ell',\) intersecting \(b\) and \(c\) at \(D\) and \(E\) respectively.

In \(\triangle ACE, BD \parallel CE,\) and therefore:

\[
\frac{AB}{BC} = \frac{AD}{DE} \quad \text{(1)}
\]

Both quadrilaterals \(ABD'C'\) and \(DEC'B'\) are parallelograms, and therefore

\[
AD = A'B', \quad DE = B'C' \quad \text{(2)}
\]

From (1) and (2),

\[
\frac{AB}{BC} = \frac{A'B'}{B'C'}
\]

**Let's try**

In the above proof, we drew a straight line parallel to \(\ell'\) and passing through point \(A,\) then applied the theorem on triangles and ratios on page 118. Try to think of other ways to prove the above theorem on parallel lines and proportions.
Example 1

In the diagram on the right, if $\ell$, $m$, and $n$ are parallel, find the value of $x$.

**Answer**

Since $\ell$, $m$, and $n$ are parallel,

\[
6 : 15 = x : 20
\]

\[
120 = 15x
\]

\[
x = 8
\]

**Check**

In the diagram on the right, if $\ell$, $m$, and $n$ are parallel, find the value of $x$.

Problem 1

Find the values of $x$ in each of the following diagrams, if $\ell$, $m$, and $n$ are parallel.

**Window on math**

**Using line segments to represent products and quotients**

We can represent the product or quotient of $a$ and $b$ by using line segments, as shown on the right. Try to see why this works.
Using properties of parallel lines and proportions, we can divide a line segment AB into three equal parts as follows.

1. From A draw a ray AX.
2. On AX, mark three equally spaced points P, Q, and R in order from A, and join points R and B.
3. From points P and Q draw lines parallel to RB, intersecting AB at S and T respectively.

Problem 2 Explain why the above procedure results in S and T dividing line segment AB into three equal parts.

Problem 3 Given line segment AB, find the point P which divides AB in a 3:2 ratio.

Example 2 In \( \triangle ABC \), if the bisector of \( \angle A \) intersects side BC at point D, then

\[ \frac{AB}{AC} = \frac{BD}{DC} \]

Prove this.

Hint Draw a line parallel to AD passing through point C, and let this intersect the extension of BA at point E. Then \( \triangle ACE \) is an isosceles triangle, and \( AB:AC \) is equal to \( AB:AE \).

Problem 4 Give a proof for Example 2.
1. In the diagram on the right, A, B, and C are points on a circle O, and BC is the diameter. If \( \angle ABC = 60^\circ \), and the intersections of the bisector of \( \angle ABC \) with the chord AC and circle O are D and E respectively, then \( \triangle ABC \sim \triangle EDC \). Prove this.

2. In the diagram on the right, if \( \angle ACB = \angle CDB = 90^\circ \) answer the following questions.

   ① Using the fact that \( \triangle ACD \), \( \triangle CBD \), and \( \triangle ABC \) are all similar, find the lengths of AD, BD, and CD.

   ② Using the result of (1), calculate the areas of \( \triangle ACD \) and \( \triangle CBD \). Then find the ratio of the areas of \( \triangle ACD \) and \( \triangle CBD \).

---

**Let's investigate!**

On page 113 we considered how to find the height of a school building from a scale drawing. Try this out, to find the height of your school building.

- Try making your own tool for finding the angle of the line of sight to the top of the building.
- Try to think of other ways of measuring the height of the building.
Window on math — Point moving parallel to a side

From point P on side AB of \( \triangle ABC \), draw a line parallel to side AC. When this intersects side BC, draw a line from the intersection, parallel to side AB. By repeating this process, you will return to the original point.

In each of the triangles below, check that the path returns to point P.

1. Draw more triangles yourself, and check with them too.

2. What is the total length of the path drawn?
Basic Exercises

1. **Triangles and Proportions**
   
   Find the values of $x$ and $y$ in each of the following if $DE \parallel BC$.
   
   - **Example 1**
     
     $\begin{align*}
     & \text{Diagram} \\
     & A \quad x \\
     & D \quad 6 \\
     & B \quad 3 \\
     & C \quad 12
     \end{align*}$

   - **Example 2**
     
     $\begin{align*}
     & \text{Diagram} \\
     & D \quad 6 \\
     & E \quad 5 \\
     & A \quad 4 \\
     & B \quad 10 \\
     & C
     \end{align*}$

2. **Triangle Midpoint Theorem**
   
   In the diagram on the right, $E$ is the midpoint of $BD$. Let $Q$ be the intersection of the extension of $PE$ and $CD$.
   
   If $AD \parallel BC$, find the values of $x$ and $y$.

   $\begin{align*}
   & \text{Diagram} \\
   & A \quad 6 \\
   & B \quad 4 \\
   & C \quad 3 \\
   & D \quad 6 \\
   & Q \quad 3 \\
   & B \quad 12
   \end{align*}$

3. **Parallel lines and Proportions**
   
   Find the values of $x$ in each of the following if $\ell$, $m$, and $n$.

   - **Example 1**
     
     $\begin{align*}
     & \text{Diagram} \\
     & \ell \quad 10 \\
     & m \quad 6 \\
     & n \quad 9
     \end{align*}$

   - **Example 2**
     
     $\begin{align*}
     & \text{Diagram} \\
     & \ell \quad 6 \\
     & m \quad 7 \\
     & n \quad 15
     \end{align*}$
1. In each of the following diagrams, find the value of $x$.

   ![Diagram 1]

   1. $\triangle ABC$ with sides $AB = 6$, $BC = 9$, and $AC = 4$.
   2. $\triangle DEF$ with sides $DE = 3$, $EF = 6$, and $DF = 4$.

2. In the regular pentagon $ABCDE$ shown in the diagram on the right, draw diagonals $AC$, $AD$, and $CE$. Let $P$ be the intersection of $AD$ and $CE$.
   In this case, give all triangles which are similar to $\triangle ACD$. Also give all triangles which are similar to $\triangle PDE$.

3. In the circle shown in the diagram on the right, two chords $AB$ and $CD$ are drawn, and their extensions intersect at $P$.
   In this case, prove that $\triangle ADP \sim \triangle CBP$.

4. In the quadrilateral $ABCD$ in the diagram on the right, the midpoints of sides $AD$ and $BC$ are $P$ and $Q$ respectively. The midpoints of diagonals $AC$ and $BD$ are $R$ and $S$ respectively, then the quadrilateral $PSQR$ is a parallelogram. Prove this.
Cutting out a square

We are going to cut out a square from a triangular piece of paper. With one side of the square on the edge of the triangle, we want the square to be as large as possible. How should we cut it?

Try to find a method, using the following diagram as a hint.

The diameter of the semicircle below is AB. Draw a square within the semicircle with two corners on the line segment AB and two corners on the semicircle.
The diagram below shows a right triangle formed by three squares. We will investigate the relationship of the areas of these squares.

In the diagram above, compare the total area of parts (a) to (e) with the area of (f).

Cut out the parts (a) to (e) on page 208, and arrange them on the square (f).
1) In figure (1) below, let the areas of the three squares be P, Q, and R. Find the values of P, Q, and R, and enter them in the table on the right.

<table>
<thead>
<tr>
<th></th>
<th>P</th>
<th>Q</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

2) Draw more right triangles with different sides, and find the areas of the corresponding three squares in the same way.

3) From what you have found so far, what do you think is the relationship of the areas of the three squares?
The Pythagorean Theorem

As shown in the diagram on the right, if \( P \), \( Q \), and \( R \) represent the areas of the squares, from what you found on the previous page we can expect that the following holds:

\[
P + Q = R \quad \text{........................ (1)}
\]

If the lengths of the sides \( BC \), \( CA \), and \( AB \) of the right triangle are \( a \), \( b \), and \( c \), respectively then

\[
P = a^2, \quad Q = b^2, \quad R = c^2
\]

We can rewrite equation (1) in terms of the lengths of the sides as:

\[
a^2 + b^2 = c^2
\]

**Problem 1**  In the above diagram, check that \( a^2 + b^2 = c^2 \) holds.

The relation we found above is true for any right triangle.

---

**The Pythagorean Theorem**

**Theorem**

If the sides adjacent to the right angle of a right triangle are of length \( a \) and \( b \), and the length of the hypotenuse is \( c \), then the following relation holds:

\[
a^2 + b^2 = c^2
\]

**Note**  We can also write \( a^2 + b^2 = c^2 \) as \( BC^2 + CA^2 = AB^2 \).

This theorem is called the **Pythagorean Theorem** (or Pythagoras' Theorem), after the Greek mathematician Pythagoras (approximately 572 to 492 BC). But the theorem appears to have been known from ancient Egyptian times.
Let's now prove the Pythagorean theorem.

Proof

We draw right triangle ABC, with three sides of length $a$, $b$, and $c$, and $\angle C = 90^\circ$. We then draw three other congruent triangles, around the four sides of a square with sides of length $c$, as shown in the diagram on the right. The outer perimeter forms a square of side $a + b$.

We can write the area relationship as:

\[
\text{(area of square of side } c) = \text{(area of outer square)} - \text{(area of } \triangle ABC) \times 4
\]

Thus:

\[
c^2 = (a + b)^2 - \frac{1}{2}ab \times 4
\]

\[
= (a^2 + 2ab + b^2) - 2ab
\]

\[
= a^2 + b^2
\]

And therefore

\[
c^2 = a^2 + b^2
\]

There are many different ways of proving the Pythagorean theorem.

**Problem 2** Right triangle $ABE$ with $\angle E = 90^\circ$, and three congruent triangles are arranged as shown in the diagram on the right. From this diagram, prove that $a^2 + b^2 = c^2$.

What is the length of a side of the small square in the center?
Now we'll use the Pythagorean theorem to find the lengths of sides of right triangles.

**Example 1**

For each of the right triangles below, find the length of side BC.

1. Since BC is the hypotenuse,
   \[ 8^2 + 6^2 = x^2 \]
   \[ x^2 = 100 \]
   Since \( x > 0 \),
   \[ x = 10 \]
   Answer 10 cm

2. Since the hypotenuse is 4 cm,
   \[ 2^2 + x^2 = 4^2 \]
   \[ x^2 = 12 \]
   Since \( x > 0 \),
   \[ x = 2\sqrt{3} \]
   Answer \( 2\sqrt{3} \) cm

**Check 1**

In the following diagrams, find the value of \( x \).

1. \[ \sqrt{17^2 - 15^2} = x \]
2. \[ \sqrt{5^2 - 5^2} = x \]

**Problem 3**

In a right triangle, let the sides adjacent to the right angle be \( a \) and \( b \), and the hypotenuse be \( c \). Complete the table below.

<table>
<thead>
<tr>
<th></th>
<th>( a )</th>
<th>( b )</th>
<th>( c )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>4</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>5</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>21</td>
<td>29</td>
<td></td>
</tr>
</tbody>
</table>

Is there an easy way to calculate \( 29^2 - 21^2 \)?
2 Inverse of the Pythagorean Theorem

On the right, draw \( \triangle ABC \), with sides \( AB = 5 \text{ cm}, BC = 4 \text{ cm}, \text{ and } CA = 3 \text{ cm} \).

What sort of triangle is \( \triangle ABC \)?

When you construct \( \triangle ABC \) as in the trigger question above, the lengths of the three sides satisfy the relation \( 3^2 + 4^2 = 5^2 \). Let's see if such a triangle is a right triangle.

To check this, draw a right triangle \( \triangle DEF \), with

\[
EF = 4 \text{ cm}, \quad FD = 3 \text{ cm}, \quad \angle F = 90^\circ
\]

Then let's consider whether this triangle is congruent to \( \triangle ABC \).

**Problem 1** Answer the following questions.

1. In the right triangle \( \triangle DEF \) above, how many centimeters long is side DE?
2. Show that \( \triangle ABC \cong \triangle DEF \)

From the result of Problem 1 we can say that \( \angle C = 90^\circ \), and thus \( \triangle ABC \) is a right triangle.

Thus the following theorem always holds.

**Inverse of the Pythagorean Theorem**

**Theorem**

If the lengths of the sides of a triangle are \( a, b, \) and \( c \), and the following holds:

\[
a^2 + b^2 = c^2
\]

Then the triangle is a right triangle with the side of length \( c \) as hypotenuse.
Using the inverse of the Pythagorean theorem, for triangles whose three sides are known, let’s check if they are right triangles.

Example 1
If the lengths of the sides of a triangle are 5, 12, and 13, can we say this is a right triangle?

Hint
We can just check if the lengths of the sides $a$, $b$, and $c$, satisfy $a^2 + b^2 = c^2$. In this case $c$ must be the longest side.

Answer
Consider $a = 5$, $b = 12$, and $c = 13$.

\[
\begin{align*}
  a^2 + b^2 &= 5^2 + 12^2 = 25 + 144 = 169 \\
  c^2 &= 13^2 = 169
\end{align*}
\]

Therefore the relation $5^2 + 12^2 = 13^2$ holds.

Answer
Yes, this is a right triangle.

Check 1
If the lengths of the sides of a triangle are 4 cm, 5 cm, and 6 cm, can we say this is a right triangle?

Problem 2
Each of the following is a set of sides of a triangle. Which are right triangles?

(a) \(\sqrt{7} \text{ cm}, \sqrt{11} \text{ cm}, 3\sqrt{2} \text{ cm}\)

(b) 9 cm, 15 cm, 17 cm

(c) 24 cm, 25 cm, 7 cm

Window on math - The “rope-stretchers”

Some 5000 years ago, the pyramids in Egypt were built with bases which are square to a very high precision. It is said that at the time people called “rope-stretchers” would take ropes in the proportions 3:4:5 and stretch them to mark an accurate right angle.
Basic Exercises

The Pythagorean theorem

1. In each right triangle in the following diagram, find the length of side AB.

   ![Diagram of right triangles with sides 6 cm, 3 cm, and 2 cm.]

   - In each right triangle, the length of side AB can be found using the Pythagorean theorem: $a^2 + b^2 = c^2$.

Inverse of the Pythagorean theorem

2. Each of the following is a set of sides of a triangle. Which are right triangles?

   - (a) 5 cm, 7 cm, 9 cm
   - (b) $\sqrt{2}$ cm, $\sqrt{3}$ cm, $\sqrt{5}$ cm
   - (c) 0.6 m, 0.8 m, 1 m
   - (d) 1 cm, 2 cm, $\sqrt{3}$ cm

Window on math - Drawing square root lengths

In the following diagram, if AB = 1, then these constructions produce line segments AC, AD, and so on whose lengths are $\sqrt{2}$, $\sqrt{3}$, and so on...

1. Try drawing the constructions above.

2. Explain why the lengths of AC, AD, ... are $\sqrt{2}$, $\sqrt{3}$, and so on.

Try making a ruler that can measure $\sqrt{2}$ cm, $\sqrt{3}$ cm, and so on...
Applications of the Pythagorean Theorem

Applications to Triangles and Quadrilaterals

We can use the Pythagorean theorem to find the length of a side, by drawing a right triangle within the figure.

Example 1 Find the diagonal of a square of side 3 cm.

**Hint** Drawing the diagonal creates an isosceles right triangle.

**Answer** If the length of the diagonal is \( x \) cm, then

\[
 x^2 = 3^2 + 3^2 = 18
\]

since \( x > 0 \),

\[
 x = \sqrt{18} = 3\sqrt{2}
\]

Answer \( 3\sqrt{2} \) cm

Check 1 Find the diagonal of a square of side 7 cm.

Example 2 Find the height of an equilateral triangle of side 2 cm.

**Hint** As shown in the diagram on the right we drop a perpendicular from point A to the side BC intersecting BC at point D. Point D is also the midpoint of BC. Then we can use the Pythagorean theorem on the right triangle ABD.

**Answer** Let \( AD = h \) cm, since \( BD = 1 \) cm

\[
1^2 + h^2 = 2^2
\]

Therefore \( h^2 = 3 \)

Since \( h > 0 \),

\[
h = \sqrt{3}
\]

Answer \( \sqrt{3} \) cm

Check 2 Find the height of an equilateral triangle of side 6 cm.
Problem 1: For an equilateral triangle of side $a$, answer the following questions.

1. Express the height in terms of $a$.
2. If the area is $S$, show that $S = \frac{\sqrt{3}}{4}a^2$.

Problem 2: Answer the following questions.

1. If the lengths of two adjacent sides of a rectangle are $a$ and $b$, express the length of the diagonal $x$ in terms of $a$ and $b$.

2. Find the height $AH$ and area of the isosceles triangle $ABC$ in the diagram on the right.

For right triangles with angles 90°, 30°, and 60° or 90°, 45°, and 45° the following relations determine the lengths of the sides.

Note: The two right triangles shown in pink above are used for set squares.

Check 3: In the diagram on the right, find the values of $x$ and $y$.
Applications to Circles

We can apply the Pythagorean theorem to circles, to find the lengths of chords and tangents.

**Example 3**

In a circle center O of radius 7 cm, find the length of chord AB at a distance of 2 cm from the center.

**Hint**

As in the diagram on the right, if a perpendicular is drawn from O to intersect AB at H, then H is the midpoint of AB. OH = 2 cm.

**Answer**

In the diagram on the right, if \( AH = x \) cm, then since \( \triangle OAH \) is a right triangle:

\[
x^2 + 2^2 = 7^2
\]

\[
x^2 = 45
\]

Since \( x > 0 \),

\[
x = \sqrt{45} = 3\sqrt{5}
\]

Since the length of chord AB is twice the length of AH,

\[
AB = 2 \times 3\sqrt{5} = 6\sqrt{5}
\]

Answer: \( 6\sqrt{5} \) cm

**Check**

The radius of a circle is 6 cm. The circle has a chord 8 cm long. Find the distance between the center and the chord.

**Problem 3**

In the diagram on the right, AP is a tangent to a circle O, with point of contact at P.

1. If we draw the radius OP, then \( \triangle APO \) is a right triangle. Explain why this is so.

2. If the radius of the circle center O is 3 cm, and the length of the line segment OA is 8 cm, find the length of line segment AP.

---

140  6 — The Pythagorean Theorem
Distance Between Two Points

If we know the coordinates of two points, we can apply the Pythagorean theorem to find the distance between them.

**Example 4** Find the distance between the two points A (5, 4) and B (-1, -3).

**Hint** Construct a right triangle with AB as the hypotenuse, and the other two sides parallel to the coordinate axes.

**Answer** As in the diagram on the right, construct right triangle ABC. We have:

- \( BC = 5 - (-1) = 6 \)
- \( AC = 4 - (-3) = 7 \)

Thus AB = d,

\[
 d^2 = 6^2 + 7^2 = 85
\]

Since d > 0, \( d = \sqrt{85} \)

**Check 5** Find the distance between the two points A (4, 3) and B (2, -1).

**Problem 4** For the following two cases, find the distance between the two points A and B.

1. Points A and B in the diagram on the right.
2. A (3, 2) and B (-3, 4)
Applications to Three-Dimensional Figures

Diagonal of a Rectangular Parallelepiped

Example 1
In the diagram on the right, if the rectangular parallelepiped has GH = 2 cm, FG = 4 cm, and BF = 3 cm, find the length of the diagonal BH.

Hint
Drawing the diagonal FH of the bottom surface forms a right triangle △ BFH with the diagonal BH as hypotenuse.

Answer
Since △ FGH is a right triangle,
\[ FH^2 = 4^2 + 2^2 \] (1)
And since △ BFH is also a right triangle,
\[ BH^2 = FH^2 + 3^2 \] (2)

From (1) and (2), \[ BH = \sqrt{29} \] cm

Note
AG, CE, and DP are also diagonals of the rectangular parallelepiped, and they are all the same length.

Check
Find the length of the diagonal of a rectangular parallelepiped of length 3 cm, width 5 cm, and height 4 cm.

Problem 1
The length of the diagonal of a rectangular parallelepiped whose length, width, and height are \(a\), \(b\), and \(c\) respectively is \(\sqrt{a^2 + b^2 + c^2}\). Show that this is true.

Problem 2
Find the length of a diagonal of a cube of side 6 cm.

Let's try
Taking the shape of your classroom as a rectangular parallelepiped, find the length of a diagonal.

142 6 — The Pythagorean Theorem
Applications to Cones and Pyramids

Example 2

Find the volume of a cone with a base of radius 5 cm and generator of length 13 cm.

**Hint**
To find the volume we need to know the height. As shown in the diagram on the right, if we draw a line segment AO representing the height, then Δ ABO is a right triangle.

**Answer**
In the diagram above right, if \( AO = h \) cm,

\[
\begin{align*}
    h^2 + 5^2 &= 13^2 \\
    h^2 &= 144 \\
    h &= 12.
\end{align*}
\]

Therefore the volume is

\[
\frac{1}{3} \pi \times 5^2 \times 12 = 100 \pi
\]

**Answer** 100\( \pi \) cm\(^3\)

**Check 2**
Find the volume of a cone of height 4 cm, with generator of length 6 cm.

**Problem 3**
A regular square pyramid has a base of side 6 cm, and the other edges of length 5 cm. If \( H \) is the point of intersection of the diagonals of the square base, find the volume of the solid by the following procedure.

1. From the length of the diagonal AC of the square base ABCD, find the length of AH. AH is half the length of AC.

2. Since \( \Delta OAH \) is a right triangle, use the Pythagorean theorem to find the length of OH.

3. Find the volume of the regular square pyramid.
Applications to Other Problems

Let's try applying the Pythagorean theorem to various problems.

Example 1: A rectangular parallelepiped has a length, width, and height of 4 cm, 5 cm, 3 cm, respectively. The diagram on the right shows a string stretched from vertex A over edge BF to vertex G. What is the length of the string at its shortest?

Try drawing the shortest possible length for the string on the unfolding diagram below.

The shortest line joining two points is a straight line segment, isn't it?

Problem 1: In Example 1, find the shortest possible length of the string.

Let's try!

In the problem above, to stretch a string from vertex A to vertex G we might also consider passing over a different edge from BF. Which way should the string pass for its length to be a minimum? Try drawing in the possibilities on the unfolding shown above. Find the length in these other cases.

144 — The Pythagorean Theorem
Sometimes when we apply the Pythagorean theorem to a problem it results in an equation.

**Example 2**

Rectangle ABCD has AB = 4 cm and BC = 6 cm. We fold this rectangle as shown in the diagram on the right, so that the vertex D comes to the midpoint M of side BC. In this case, find the length of CF.

In the diagram above, mark line segments of equal length.

**Hint**

Since \( \triangle EMF \) is created by folding \( \triangle EDF \):

\[
MF = DF
\]

If \( CF = x \) cm, then

\[
MF = (4 - x) \text{ cm}
\]

**Problem 2**

In the problem above, apply the Pythagorean theorem to right triangle FMC to obtain an equation, and find the length of CD.

Let's try applying the Pythagorean theorem to problems in real life.

**Problem 3**

From a map showing the Hakkoda ropeway, we can see that from the bottom station to the mountaintop station is a horizontal distance of approximately 2.4 km, while the vertical ascent is approximately 0.7 km. Considering the ropeway as a straight line, what is the approximate total length in kilometers?
Basic Exercises

Applications to two-dimensional figures

Find the following values.

1. The length of the diagonal of a square of side 10 cm
2. The height of an equilateral triangle of side 12 cm
3. The length of chord AB of the circle center O on the right
4. The distance between the points A (−3, 3) and B (−1, −3)

Ratio of sides of special right triangles

In the following diagrams, find the values of $x$ and $y$.

Applications to three-dimensional figures

Find the following values.

1. The length of a diagonal of a rectangular parallelepiped of length 3 cm, width 4 cm, and height 12 cm
2. The volume of a cone with a base of radius 6 cm and generator of length 10 cm

146 6 — The Pythagorean Theorem
1. The diagram on the right shows a piece of paper tape of width 4 cm, folded at the line segment AB. If $\angle ACB = 45^\circ$, find the area of the overlapping area, $\triangle ACB$.

2. In the diagram on the right, $A$ and $B$ are points on the graph of $y = 2x^2$, whose x-coordinates are $-1$ and $2$ respectively. Find the length of the line segment $AB$.

3. The lateral surface of a cone can be flattened to a semicircle of radius 10 cm. Find the height and volume of this cone.

4. A pair of set squares have sides in the relationship shown in the diagram on the right.
   1. In the diagram on the right, if $AC = 10$ cm, find the lengths of the other sides.
   2. If each of the set squares is placed on its hypotenuse as base, the height is the same. Explain why this is so.

5. As shown in the diagram on the right, an equilateral triangle is drawn on each side of a right triangle. In this case, what is the relation governing the areas of the equilateral triangles $P$, $Q$, and $R$?
Chapter Summary Problems B

1. In right triangle ABC, AB is 8 cm longer than BC, and BC is 1 cm longer than CA. Find the length of the hypotenuse.

2. Semicircles are drawn with a diameter on each of the sides of right triangle ABC, as shown in the diagram on the right. Prove that in this case the sum of the areas of the shaded portions is equal to the area of the right triangle ABC.

3. Find the area of \( \triangle ABC \) shown in the diagram on the right, using steps 1 to 3.
   1. Let \( AH = h \), and \( BH = x \), then applying the Pythagorean theorem to \( \triangle ABH \) and \( \triangle ACH \), find an expression in each case for \( h^2 \) in terms of \( x \).
   2. Eliminate \( h^2 \) from the equations you found in step 1, to find the value of \( x \).
   3. Find the value of \( h \), and then the area of \( \triangle ABC \).

Let's investigate:

There are many proofs of the Pythagorean theorem.

- Look for more proofs in books or on the Internet.
- When you find a proof, write it out on a large sheet of paper, and give a presentation to your friends.
- Choose one proof, and think of a way to make a model illustrating it.
Let's prove the Pythagorean theorem by constructing models.

As shown on the right, make four right triangles with sides $a$, $b$, and $c$, and one square of side $b-a$. We can fit these together in two ways, as follows.

Compare the areas in cases (a) and (b), and derive $a^2 + b^2 = c^2$.

As shown in (a) below, cut out a piece of card in such a way that AGDE and CFBG are squares, then separate this into two parts, and join points A and B and points C and D with pieces of rubber band. Compare the area of the square ACDB in (b) with the area of the two squares in (a), to derive the Pythagorean theorem.
This is a famous old problem.

There are three people, Arthur, Bert, and Charles, and just one of them tells the truth, and the other two are liars. From the following conversation, deduce who is the one who tells the truth.

Arthur: Bert:

I tell the truth.

Arthur is a liar. I tell the truth.

Bert is a liar. Actually I am the one who tells the truth.

Charles:

Four teams, A, B, C, and D hold a soccer competition.

Each team plays each other either once or twice.

• Team A: 4 wins, 0 losses
• Team B: 2 wins, 1 losses
• Team C: 0 wins, 3 losses

What is team D's score?
Appendix

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Finding the Catch-up Point

In the problem on page 95, if Alex and Boris start at the same time, Boris will catch up with Alex at some point down the slope.

If the slope is 40 meters long, with start point P and end point Q, consider the case with the starting times as follows.

Alex starts one second before Boris.

In this case, will Alex reach the bottom of the slope before Boris catches up with him?

We can draw graphs showing the progress of Alex and Boris as in the following figure.

Problem 1 If Alex starts one second before Boris, who will reach point Q first?
You have seen that even if Alex starts one second before Boris, Boris will catch him up before the end of the slope. How many seconds after Boris starts and how many meters from point P would he catch up?

**Problem 2** If $x$ seconds after Boris starts Alex is $y$ meters from point P, express $y$ in terms of $x$.

The $x$ and $y$ coordinates of the intersection of the parabola $y = \frac{1}{2} x^2$ and the straight line $y = 4x + 4$ represent the time and position where Boris catches up with Alex. The values of the $x$ and $y$ coordinates are the solutions to the combination of a quadratic equation and a linear equation, as the following simultaneous equations:

\[
\begin{align*}
  y &= \frac{1}{2} x^2 \\
  y &= 4x + 4
\end{align*}
\]

Therefore, we can eliminate $y$ from the above simultaneous equations, to obtain the quadratic equation:

\[
\frac{1}{2} x^2 = 4x + 4 \quad \text{..................} \quad (1)
\]

And by solving this equation we can find the $x$ coordinate of the intersection.

**Problem 3** Use the procedure on the right to solve equation (1).

Multiply both sides of (1) by 2, to get

\[
x^2 = 8x + 8
\]

Add 16 to both sides, giving

\[
x^2 - 8x + 16 = 8 + 16
\]

Find the coordinates of the intersection of the parabola $y = x^2$ and the straight line $y = 3x + 4$ by the same method as using the equations above.
Supposing that Alex starts later than Boris, let’s use graphs or the equations you learned about on the previous page to determine whether Alex catches up with Boris.

**Problem 4** If Alex starts 1.5 seconds after Boris, answer the following questions.

1. When $x$ seconds have elapsed after Boris starts, if Alex is $y$ meters from point P, write an expression for $y$ in terms of $x$.
2. While between points P and Q, what is the positioning of Alex and Boris? That is, who is ahead?

Now consider the case when Alex starts two seconds after Boris.

In this case Alex's progress is shown by the straight line in the diagram on the right. Will Alex catch up with Boris on the slope?

**Problem 5** If Alex starts 2 seconds after Boris, answer the following questions.

1. When $x$ seconds have elapsed after Boris starts, if Alex is $y$ meters from point P, write an expression for $y$ in terms of $x$.
2. Will Alex pass Boris?

**Problem 6** If Alex starts 2.5 seconds after Boris, answer the following questions.

1. In the above diagram, draw a line showing Alex's progress.
2. Will Alex catch up with Boris on the slope?
3. Consider whether Alex will catch up with Boris, using equations. What are the solutions to the equations?
Using Similar Figures to Investigate Properties of Circles

As shown in the diagram on the right, two straight lines are drawn through point P in the interior of a circle, each intersecting the circle at points A, B and C, D respectively.

In this case, the following holds:

\[ PA \times PB = PC \times PD \]

We can prove this as follows.

**Proof**

From Example 3 on page 112:

\[ \triangle AC \sim \triangle DBP \]

Therefore, since the ratios of corresponding sides of similar triangles are equal, we have:

\[ \frac{PA}{PD} = \frac{PC}{PB} \]

In other words

\[ PA \times PB = PC \times PD \]

**Problem 1**

In the diagram on the right, one more line is drawn through point P in the diagram for the problem above, intersecting the circle at points E and F.

In this case, what is \( PE \times PF \) equal to?

**Properties of proportions (page 106)**

If \( a : b = m : n \) then

\[ an = bm \]

Further Topics 155
Next, let's consider the similar case as in problem 3 on page 126, where two straight lines are drawn through point P external to a circle each intersect the circle at points A, B and C, D respectively.

**Problem 2** In the diagram on the right, prove that the following holds:

\[ PA \times PB = PC \times PD \]

**Problem 3** In the diagram for Problem 2, draw one more straight line passing through P, and intersecting the circle at two points; let these points be E and F. In this case, what is PE \times PF equal to?

From what we have learned so far, you should understand the following.

For any circle, if we fix a point P inside or outside the circle, and draw a line passing through P to intersect the circle at points Q and R, then \( PQ \times PR \) has a constant value.
Using what you learned on the previous page, you can draw a line segment of length $\sqrt{6}$, for example, using the following steps 1 to 4.

1. Express 6 as the product of two numbers, such as 2 and 3.
2. Draw a circle whose diameter is the sum of the two numbers you chose in step 1. In other words, since $2 + 3 = 5$, draw a circle of diameter 5.
3. At point P that divides the diameter into the ratio of the two numbers, that is, 2:3, draw a line perpendicular to the diameter, and let an intersection of this perpendicular line with the circle circumference be point Q.
4. Then $PQ = \sqrt{6}$.

![Diagram](image.png)

**Problem 4** In the diagram above, explain why the length of PQ is $\sqrt{6}$.

**Problem 5** Express 6 as the product of a different pair of numbers, and use the above method to construct a length of $\sqrt{6}$ on the above diagram.

**Problem 6** Using the construction considered above, given the rectangle on the right, it is possible to construct a square with an area equal to that of the rectangle. Try to discover how to do this.
There are two similar triangles, and the ratio of similarity of the smaller triangle to the larger triangle is 1:2. In this case, we can divide the larger triangle into four copies of the smaller triangle.

What about the case in which we have two quadrilaterals with a similarity ratio of 1:2?

In the above, we can divide the larger quadrilateral into two triangles by a diagonal, as shown in the diagram. Each of these triangles can be divided into four smaller similar triangles. By combining these triangles in pairs, we can make four of the smaller quadrilaterals.

Problem 1: Draw two quadrilaterals with a similarity ratio of 1:2 on a piece of paper, then cut up the large quadrilateral as described above. Next check that you can reassemble these to make four congruent copies of the similar smaller quadrilateral.
Let's look at the relation between the similarity ratio of two similar triangles and the ratio of their areas.

**Problem 2**
In the diagram on the right,
\[ \triangle ABC \sim \triangle A'B'C' \]
and the similarity ratio of \( \triangle ABC \)
and \( \triangle A'B'C' \) is 1:2.

1. Find the lengths of \( B'C' \)
and \( A'H' \).
2. Find the areas of each of \( \triangle ABC \) and \( \triangle A'B'C' \).
3. Find the ratio of the areas of \( \triangle ABC \) and \( \triangle A'B'C' \).

**Problem 3**
In the two circles shown on the right, what is the ratio of their circumferences? And what is the ratio of their areas?

For similar plane figures, the following holds true:

For similar plane figures, the ratio of their perimeters is equal to the similarity ratio, and
the ratio of their areas is equal to the square of the similarity ratio.

In other words, if the similarity ratio is \( m:n \), the following hold:

\[
\text{Ratio of perimeters: } \frac{m}{n} \\
\text{Ratio of areas: } \frac{m^2}{n^2}
\]

Let's look at problems we can solve using the relation between the similarity ratio and the ratio of the areas.

**Problem 4**
Two similar figures P and Q have a similarity ratio of 3:4.

1. Find the ratio of their perimeters.
2. If the area of P is 36 cm\(^2\), find the area of Q.
Problem 5 In the diagram on the right, points P, Q, and R divide the side AB of \( \triangle ABC \) into four equal parts. The line segments through these points are all parallel to side BC. If the area of \( a \) is \( a \), find expressions for the areas \( b \), \( c \), and \( d \) in terms of \( a \).

Problem 6 The diagram on the right, shows two circles with a common center. The radius of the outer circle is 2 cm. Let the region between the two circles be \( a \), and the region inside the smaller circle be \( b \).

If the areas of \( a \) and \( b \) are equal, what is the radius in centimeters of the inner circle?

Surface Area and Volume Ratios of Similar Solids

We can also consider similar three-dimensional figures. If we scale a three-dimensional figure up or down at a fixed ratio, without changing the shape, this also gives a similar figure.

Let's look at a simple case of the ratios of surface area and volume of similar solids.

Trigger If we put together a number of identical cubic blocks, as in the diagram on the right, we can make cubes P and Q.

What is the ratio of the surface areas of cubes P and Q?

What is the ratio of their volumes?
Problem 7  The two square pyramids P and Q below are similar, with a similarity ratio of 1:2.

1) Find the surface areas of the two square pyramids.
2) Find the volumes of the two square pyramids.
3) Find the ratios of the surface areas and the volumes of the two square pyramids.

For similar solid figures, the following holds true:

For similar solid figures, the ratio of their surface areas is equal to the square of the similarity ratio, and the ratio of their volumes is equal to the cube of the similarity ratio.

In other words, if the similarity ratio is $m:n$, the following hold:

Ratio of surface areas: $\frac{m^2}{n^2}$
Ratio of volumes: $\frac{m^3}{n^3}$

Problem 8  In the diagram on the right, a conical container is filled with water to a depth of 4cm.

1) Find the capacity of the conical container.
2) The part of the container occupied by the water and the container are similar shapes. Find the similarity ratio.
3) Find the volume of water in the container.

Let’s try!

If the container in Problem 8 contains half of its volume of water, how deep is the water in centimeters? Calculate this to two decimal places using a calculator.
As in the diagram below, joining the midpoints of the sides of $\triangle ABC$ in sequence constructs $\triangle A_1B_1C_1$. Then joining the midpoints of the sides of $\triangle A_1B_1C_1$ in sequence constructs $\triangle A_2B_2C_2$. If we continue this process, to construct $\triangle A_3B_3C_3$, $\triangle A_4B_4C_4$, and so on, what can we say about $\triangle A_nB_nC_n$? Let's draw a diagram to find out.

The line segment joining a vertex of a triangle to the midpoint of the opposite side is called a median. In the previous question, $AA_1$, $BB_1$, and $CC_1$ are the medians of $\triangle ABC$.

Problem 1. In the previous question, we can prove that the midpoint $A_2$ of $B_1C_1$ lies on the median $AA_1$ of $\triangle ABC$, by the following method.

1. Show that the quadrilateral $AC_1A_1B_1$ is a parallelogram.
2. From the properties of a parallelogram, show that $A_2$ is the intersection of $AA_1$ and $B_1C_1$. 

Answer
From Problem 1, we can see that the vertices $A_1, A_2, A_3, \ldots$ of the triangles constructed in the question all lie on the median $AA_1$ of $\triangle ABC$. Similarly, we can see that the vertices $B_1, B_2, B_3, \ldots$ lie on the median $BB_1$, and the vertices $C_1, C_2, C_3, \ldots$ lie on the median $CC_1$.

For $\triangle A_1B_1C_1$ constructed in the question, as $n$ gets larger and larger, the three vertices of the triangle get closer and closer to a single point. This fact, together with what we learned above, suggests that the three medians $AA_1, BB_1$, and $CC_1$ of $\triangle ABC$ must intersect at a single point $G$.

The point of intersection of the three medians of a triangle is called the centroid of the triangle.

**Problem 2**  
In the diagram above, join points $A_1$ and $B_1$, and prove that $GA : GA_1 = 2 : 1$.

In the same way as Problem 2, we can show that

$GB : GB_1 = 2 : 1$
$GC : GC_1 = 2 : 1$

And therefore that

$GA : GA_1 = GB : GB_1 = GC : GC_1 = 2 : 1$

In other words, the centroid of a triangle divides each median in the ratio 2:1.

Each of the triangles constructed in the question is in a position of similarity with respect to the centroid as center of similarity.

**Problem 3**  
In your notebook, draw a triangle, and construct the centroid of the triangle.

**Problem 4**  
If $G$ is the centroid of $\triangle ABC$, show that:

$\triangle ABG = \triangle BCG = \triangle CAG$
Slope and Height of a Roof

As shown in the illustration below, suppose we want to build a house with the roof sloping at 10° to the horizontal. What will the height of the roof be in this case?

In mountain regions of high snowfall, we often see houses with steeply sloping roofs. This is to prevent the roof collapsing under the weight of snow.

If the roof of the house shown above is sloping at an angle of 20°, 30°, 40°, 50°, or 60°, what will the height of the roof be?
The diagram on the lower-right side of this page shows a sequence of right triangles, where \( AC = 10 \text{ cm} \), and the value of \( \angle BAC \) is 10°, 20°, ... and so on up to 60°. For each of these triangles, read off the length of BC, and calculate the value of \( \frac{BC}{AC} \).

If the angle is increased 2 times, 3 times and so on, does the length of BC increase 2 times, 3 times and so on?

Using the values in the table above, find the height of the roof portion of the house on the opposite page when the slope of the roof is 20°, 30°, 40°, 50°, or 60°.

Problem 2: What is the approximate slope in degrees of the roof of the house on the left?

If we know the value of \( \frac{BC}{AC} \) for different values of \( \angle BAC \), we can work this out without drawing a right triangle.

Problem 3: In the diagram on the left, a lighthouse is 80 meters tall. Looking up at the lighthouse from a boat at point P, the angle is 20 degrees. Find the distance from the boat to the lighthouse.
Discovering Regular Patterns

Suppose we take a cube, and color all of the faces, then cut this cube into smaller cubes, each of side 1 cm. Let's look at these small cubes, and how many faces of them are colored.

If we cut up a cube of side 3 cm, how many small cubes do we get which are colored on two faces?

There are four cases as shown below for the number of colored faces of a small cube, that is, with zero, one, two, or three faces colored. These are respectively the cases shown as 0, 1, 2, 3 below.

Problem 1 For each of the cubes 0 to 3, what part of the large cube do they come from?

Problem 2 For the cube of side 3 cm we investigated in the previous question, how many of cubes 0, 1, and 3 are there?
If we change the length of a side of the large cube, how does this affect the number of cubes 0 to 3?

Let’s look at cubes 3 and 2.

Problem 3 When the side of the large cube is 4 cm, how many of cubes 3 and 2 are there? And how many are there when the side of the large cube is 5 cm? Write the answers in the following table.

<table>
<thead>
<tr>
<th>Length of side (cm)</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of cube 3</td>
<td>8</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Number of cube 2</td>
<td></td>
<td>12</td>
<td></td>
</tr>
</tbody>
</table>

Problem 4 From the results in the table above, predict how many of the 3 and 2 cubes in the case when the large cube has a side of length 6 cm.

Cube 3 can only occur at the corners of the large cube, and therefore there are eight of them, regardless of the length of a side of the large cube.

Problem 5 The number of 2 cubes is a multiple of 12. Where do the cubes 2 occur in the large cube? Consider why this is so.
Now let's look at how the number of 1 cubes varies.

**Trigger** When the side of the large cube is 3 cm, there are six of 1 cubes. This is the same number as which feature of a cube?

When the side of the large cube is 4 cm, each face of the cube has four 1 cubes, as shown in the diagram on the right.

**Problem 6** From the above, for the case when the side of the large cube is 4 cm, find the total number 1 cubes.

**Problem 7** When the side of the large cube is 5 cm, how many cube 1 are there?

Finally, let's see how the number of 0 cubes varies.

The 0 cubes are in the interior of the large cube, and when the side of the large cube is 3 cm, 4 cm, or 5 cm, the number of 0 cubes is 1, 8, and 27, respectively.

**Problem 8** When the side of the large cube is 6 cm, how many 0 cubes are there?

**Problem 9** When the side of the large cube is \( n \) cm, how many of each 0 cubes to 3 are there?

Here \( n \) is a natural number greater than or equal to 3.

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Using a Carpenter's Square

The "carpenter's square" is a tool used in woodwork, as shown in the photograph on the right. On the reverse of the square there is an outer scale (in centimeters), and an inner scale ("diagonal scale"), in which the graduations are $\sqrt{2}$ times far apart. We can use these two scales as follows.

1. Finding the size of square timber that can be cut from a log

With the diagonal scale on the diameter of a log, we can directly read off the maximum dimensions of square timber that can be cut.

2. Drawing a Regular Octagon

We can draw a regular octagon as follows.

1. Draw a square, and measure the side with the diagonal scale.
2. From each corner, mark off the same reading on the outside (centimeter) scale along both adjacent sides.
3. Join the marks as shown in the diagram on the right.

Work out why the above procedure produces a regular octagon.
Paper Sizes and Photocopier Zoom Ratios

Standard paper sizes used in Japan are the A-series and B-series of sizes; this textbook is B5 size. (Note that the A-series is an international standard, but the B-series is a JIS standard, which is not the same as the international standard B-series. For clarity, the sizes we talk about here are properly called "JIS B5" and so on.) Two sheets of B5 size together make one sheet of B4 size, and the two sizes are similar shapes.

The dimensions of B4 and B5 sizes are defined as follows.

1. B0 size is a rectangle with an area of $1.5 \text{ m}^2$.

2. If a sheet of B0 size is cut in half crosswise, each half is a sheet of B1 size, and is the a shape similar to B0.

3. In the same way, repeated cutting gives B2 size, B3 size, B4 size and so on, where all of these sizes are similar shapes.

Take a sheet of B4 paper and cut it as in 3 above, to make sheets of sizes B5, B6, and B7, then position them in positions of similarity, as shown in the diagram on the right.

If I want to enlarge up a B5 page to make a B4 copy, what magnification ratio do I need to select?
From what you learned on the previous page, derive the ratio of the height and width of B5 or B4 size by the following procedure. Let the shorter side be the height and the longer side the width.

1. Let the height of a B4 sheet of paper be 1, and the width be $x$.

2. Since the ratio of the height to width of both B5 and B4 sizes is the same, find the value of $x$.

To enlarge a B5 page to make a B4 copy, what magnification ratio do we need to select? Consider the ratio of the widths of the sheets. To reduce a B4 page to make a B5 copy, what magnification ratio do we need to select?

Next let's look at the relation between B sizes and A sizes.

Statements 2 and 3 on page 170 are also true of A sizes. And for A and B sizes with the same number, the diagonal of the A size is equal to the longer side of the B size.

From the diagram above, find the magnification ratio needed to enlarge an A4 page to B4 size.

How many times the area of an A4 sheet is the area of a B4 sheet?
The Golden Ratio

The length-to-width ratio, or aspect ratio, of a typical textbook or notepad is $\sqrt{2} : 1$. But there are many rectangles around us with other aspect ratios, for example business cards and telephone cards, or the Japanese shinsho book format.

1. For business cards, telephone cards, and books in the shinsho format, find actual examples, and measure the ratio between the longer dimension and the shorter dimension.

The rectangles you investigated above have that property that if you cut a square from the rectangle to leave another rectangle, these two rectangles are similar.

For a rectangle with this property, find the ratio of the two sides.

In the above rectangle, let the height be 2 and the width $x$; then from the relation between the ratios of corresponding lengths,

$$ \frac{2}{x} = \frac{x - 2}{2} $$

From this equation, we get the equation $x^2 - 2x - 4 = 0$. Solving this equation gives $x = 1 \pm \sqrt{5}$, and since $x > 0$ is positive, $x = 1 + \sqrt{5}$.

2. Taking $\sqrt{5} = 2.236$, find the value of $1 + \sqrt{5}$, and check that the aspect ratio of the rectangles above is approximately 5:8.
The Golden Ratio

Since the times of ancient Greece the ratio $2 : (1 + \sqrt{5})$ has been thought to be the most harmonious ratio, and is called the Golden Ratio. The golden ratio can be found in architecture, painting and sculpture. In one well-known example, in the Venus de Milo, the ratio of the heights of the parts above and below her navel is approximately the golden ratio.

The golden ratio also emerges within a regular pentagon. In the regular pentagon on the right, the angles marked with a black spot are all equal, and therefore, $\triangle DJC \sim \triangle ACJ$ and therefore the following holds:

\[ DJ : AC = JC : CD \]

If $AC = x$, $CD = 2$, check that you can obtain the same equation as on the previous page.

From this, we can see that the ratio of the length of a diagonal to the length of a side of a regular pentagon is the golden ratio.

Find out about the golden ratio from books and Internet articles on art, to see some of the places where the golden ratio occurs.

Parthenon stands (Greece)
Using Functions to Consider the Problem of Global Warming

Gases such as carbon dioxide in the atmosphere have the same effect as a greenhouse glass, increasing the air temperature. If the concentration of these gases increases there are fears that this will lead to increasing global average temperatures.

For example, if the average temperature rises by 2 degrees C, rising sea level and serious effects on forestry and agriculture, and plant and animal life are predicted.

Investigate what events have been attributed to the effects of global warming.

Here we will consider the variation in carbon dioxide concentration.

The following table shows the concentration of carbon dioxide as measured at Mauna Loa observatory in Hawaii. The values are given in parts-per-million (ppm).

<table>
<thead>
<tr>
<th></th>
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<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂ Concentration</td>
<td>317</td>
<td>320</td>
<td>326</td>
<td>331</td>
<td>339</td>
<td>346</td>
<td>354</td>
<td>361</td>
<td>369</td>
</tr>
</tbody>
</table>
Based on these figures, let's look at how the change has occurred.

2. In the diagram on the right, enter the points with the years and concentration values as coordinates. What can you say about the change in concentration?

3. Look at the increase in concentration every 10 years from 1960, and consider what kind of change this is.

The concentration of carbon dioxide was approximately 270 ppm, and did not change appreciably until the industrial revolution. However, from the nineteenth century coal and oil were used in large quantities, producing carbon dioxide, as a result of which the concentration of carbon dioxide in the atmosphere has continued to rise rapidly. If the concentration of carbon dioxide reaches double the level before the industrial revolution, or 550 ppm, it is thought that average temperatures will rise by 2°C.

Based on the figures in the chart at the bottom of the previous page, let's try to predict when the carbon dioxide concentration will exceed 550 ppm.
First we will look at the past data in the following two ways.

1. The graph you drew on the previous page shows that the growth from 1980 to 2000 was roughly linear, with an increase of 15 ppm every ten years.
   If we assume this will continue in the same way, we hypothesize that the concentration of carbon dioxide will increase at a constant rate of 15 ppm per ten years.

2. If we look at the data of the whole forty-year period from 1960, the change in each ten-year period varies as shown in the following table.

<table>
<thead>
<tr>
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<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Concentration</td>
<td>317</td>
<td>326</td>
<td>339</td>
<td>354</td>
<td>369</td>
</tr>
<tr>
<td>Increase</td>
<td>9</td>
<td>13</td>
<td>15</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>Change in increase</td>
<td>+4</td>
<td>+2</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

In the 1990s the increase did not change, but we could consider that over the forty-year period as a whole, the increase itself gradually grew, and average out the change in the increase, as shown in the following table. In other words, we can hypothesize that the increase amount tends to grow by 2 for every ten years.

<table>
<thead>
<tr>
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<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Concentration</td>
<td>317</td>
<td>327</td>
<td>339</td>
<td>353</td>
<td>369</td>
</tr>
<tr>
<td>Increase</td>
<td>10</td>
<td>12</td>
<td>14</td>
<td>16</td>
<td>16</td>
</tr>
<tr>
<td>Change in increase</td>
<td>+2</td>
<td>+2</td>
<td>+2</td>
<td>+2</td>
<td>+2</td>
</tr>
</tbody>
</table>

Using these two hypotheses, let's investigate when the concentration of carbon dioxide will exceed 550 ppm.

For the two hypotheses above, complete the following table and the graphs on the next page.
In each case, let's predict when the concentration of carbon dioxide will exceed 550 ppm.

1. In case 1, taking the concentration as \( y \) ppm, \( x \) years after 2000, express \( y \) in terms of \( x \), and find the solution from this expression.

2. In case 2, read off the value from the graph above.

In both the cases we considered above, we simply extrapolated past data into the future. The Intergovernmental Panel on Climate Change (IPCC) has considered various scenarios, and predicted a concentration of carbon dioxide for the year 2100 of between 540 ppm and 970 ppm. Where in this interval the value actually lies depends on the efforts of humanity.

6. Discuss whether there are things we ourselves can do to reduce the concentration of carbon dioxide.
Measuring Heights

The Edo-period mathematics text *Jinkoki* describes the following method of finding the height of a tree.

As shown in the diagram on the right, we cut a square of paper in half to form an isosceles right triangle ABC, and attach a weight to B.

Using the weight to check that BC is perpendicular to the ground, we find a place where AB is in line with the top of the tree D, as shown in the diagram. Then we measure the distance FG from this point to the tree, and add the height of the eye AF to obtain the height of the tree.

1. Work out why this method works to measure the height.

2. Make the equipment described above, and try actually measuring some trees around your school.

There are many other interesting examples given in *Jinkoki*. The illustration on the right uses the example of mice breeding, to illustrate geometric progression.

In the book, this example immediately follows the tree example shown above.
How Far Can We See?

The photograph below was taken from the observation deck at the top of Yokohama Landmark Tower. How far do you think we can see from here?

In the diagram on the right, if P is the position of the observation deck, O is the center of the earth, then the distance we can see from the observation deck is the length PA to the point of contact A when we draw a tangent from P to the circle center O.

1. The radius of the earth is approximately 6378 km, and the height of the observation deck of Yokohama Landmark Tower is 273 m. Using these values, use the Pythagorean theorem to find the length of PA.

2. Investigate how far you can see from a tall building or mountain near your home.
Working with Origami

Fold a square of paper as in the photograph.
Now let's look at the three right triangles formed by the single-thickness part of the paper.

When we fold the square of paper ABCD, let the new positions of the vertices A and D be A' and D', and let the line PQ be the fold, and R be the intersection of A'D' and CD.

First let's consider the case of folding so that vertex A is at the midpoint of side BC.

1. Let's fold the paper so that vertex A is at the midpoint of side BC, then cut out \( \triangle PBA', \triangle ACR, \) and \( \triangle QDR. \)

What do you think is the relation between the three triangles?

How are the lengths related? Are these in positions of similarity?
Let's look at the lengths of the sides of right triangle PBA'.

1. Write the length of PA' in terms of $x$.

2. In $\triangle PBA'$, use the Pythagorean theorem to derive an equation for $x$, and solve this to find the value of $x$.

3. What is the relation of the lengths of the three sides $PB$, $BA'$, and $A'P$ of $\triangle PBA'$? Using a different method from 2 above, look at the lengths of the three sides $\triangle PBA'$.

Using a different method from 2 above, draw line $AA'$, and let $E$ be the point of intersection with $PQ$. Draw a perpendicular from $E$ to $AB$, and let $F$ be the point at which it meets $AB$.

1. Find the lengths of $AF$ and $EF$. Find the length of $FP$.

2. Using the results of 1, look at the relation of the sides $PB$, $BA'$, and $A'P$ of $\triangle PBA'$.

Next, let's look at cases where point $A$ is not folded to the midpoint of side $BC$.

4. Try folding the paper so that point $A$ is at various positions, such as one-third or one-fourth of side $BC$.

In these cases, what happens to the results you obtained in steps 1 to 3?
Numerical Calculation

1. Calculate the following.
   1) $(+3) + (-11)$
   2) $(-9) + (+4)$
   3) $(-5) + (+16)$
   4) $(+6) - (+8)$
   5) $(-7) - (+7)$
   6) $0 - (-13)$
   7) $7 - 5 - 13 + 4$
   8) $2 + (-10) - (-8)$

2. Calculate the following.
   1) $(-6) \times 7$
   2) $(-24) \div (-4)$
   3) $15 \div \left(-\frac{5}{7}\right)$
   4) $(-5)^2 \times 2$
   5) $(-3) \times 4 \times (-1) \times (-7)$
   6) $12 \div (-8) \times 6$
   7) $10 - 3 \times (-7)$
   8) $7 \times (8 - 11)$

3. Answer the following questions.
   1) Compare 0, 0.5, and −1 using inequality signs.
   2) List the integers with an absolute value of less than 3 in increasing order.
   3) If $a - b < 0$, and $ab < 0$, are $a$ and $b$ positive or negative numbers?

4. In the table on the right, the heights of five people, A, B, C, D, and E are shown as differences from 170 cm, with a positive value indicating a greater height and a negative value a lesser height.

<table>
<thead>
<tr>
<th>Person</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
</tr>
</thead>
<tbody>
<tr>
<td>Height (cm)</td>
<td>+5</td>
<td>-6</td>
<td>-7</td>
<td>+2</td>
<td>+1</td>
</tr>
</tbody>
</table>

1) How tall is B?
2) By how many centimeters is the tallest person taller than the shortest person?
3) What is the average height of the five people?
5. Write algebraic expressions for the following quantities.
   1. The total weight of \( x \) items each 3 kg and \( y \) items each 2 kg.
   2. The cost of an item priced at 1000 yen, sold with a discount of \( d \% \).

6. Simplify the following.
   1. \( 4a - 3 - 5a + 6 \)
   2. \( x + 6y + 3x - 4y \)
   3. \( (2x + 5) - (6 - 8x) \)
   4. \( (7a + 4b) + (a - 3b) \)
   5. \( 4(2x + 3y) \)
   6. \( (10a - 4b) \div 2 \)
   7. \( 3(2a - b) - 2(a + 6b) \)
   8. \( \frac{x - 7}{3} + \frac{3x + 8}{5} \)
   9. \( (-7xy) \times 2y \)
   10. \( (-6a^2) \div 4ab \times (-10b) \)

7. Evaluate the following expressions when \( a = -2 \) and \( b = \frac{2}{3} \).
   1. \( 5 - a \)
   2. \( -2a^2 + 9b \)
   3. \( 5(a - 2b) - (3a - 7b) \)
   4. \( 6a^3b \div \left(-\frac{2}{5}a^2\right) \)

8. Solve each of the following equations for the variable shown in brackets.
   1. \( \ell = 2(a + b) \) \([\ell] \)
   2. \( V = \frac{1}{3}Sh \) \([h] \)

9. In four consecutive integers, if the smallest integer is a multiple of 3, show that the sum of all the integers is a multiple of 6.
   Prove algebraically that this is so.

10. Construct a problem so that a quantity can be represented by the expression \( 1000 - (2a + 5b) \).
Equations and Simultaneous Equations

11 Which of the following pairs of $x$ and $y$ values satisfy the linear equation $3x + y = 5$.

- (a) $x = -1, y = 8$
- (b) $x = 0, y = 2$
- (c) $x = 1, y = 2$
- (d) $x = 2, y = 1$

12 Solve the following equations.

- (1) $4x - 1 = 1$
- (2) $4x - 3 = x + 6$
- (3) $2x - 7 = 8x + 5$
- (4) $-4x - 6 = x - 6$
- (5) $3(3 - 2x) + 4 = x - 1$
- (6) $x - \frac{2x + 7}{9} = -1$

13 Solve the following simultaneous equations.

- (1) \[
\begin{align*}
    x + 3y &= 5 \\
    2x - y &= 3
\end{align*}
\]
- (2) \[
\begin{align*}
    3x + 2y &= 1 \\
    4x + 5y &= 13
\end{align*}
\]
- (3) \[
\begin{align*}
    2x + 5y &= 11 \\
    x &= 5 - 3y
\end{align*}
\]
- (4) \[
\begin{align*}
    y &= 4x + 2 \\
    y &= -x - 8
\end{align*}
\]

14 Find the value of $a$ if the solution of the linear equation $ax - 3 = 2x + 7$ is 5.

15 Consider a two-digit natural number. The sum of the two digits is 13. If the tens digit and units digit are interchanged the result is 27 more than the original number. Find the original number.

16 For a return journey between two points A and B the outward journey took 20 minutes. The return journey was at a speed 20 meters per minute slower than the outward journey. The return journey took 30 minutes. Assuming that both outward and return journeys were walking at the same speed, find the distance between A and B.
17 Answer the following questions.

1. \( y \) is proportional to \( x \), and when \( x = 8, y = -12 \). Write an expression for \( y \) in terms of \( x \).

2. \( y \) is inversely proportional to \( x \), and when \( x = 6, y = 12 \). Find the value of \( y \) when \( x = -9 \).

3. Find the equation of the linear function having a slope of \( -3 \), and passing through the point \( x = -2, y = 5 \).

4. The three points \( A(-2, -7), B(6, -1), C(10, \alpha) \) lie on a straight line. Find the value of \( \alpha \).

5. What is the range of \( y \) for the linear function \( y = \frac{2}{3}x + 4 \) if the range of \( x \) is \( -6 \leq x \leq 3 \)?

18 As shown in the diagram on the right, \( E \) is a point inside rectangle \( ABCD \). Point \( P \) moves from point \( A \) along the sides through \( B \) to \( C \). If the area swept out by line segment \( EP \) when point \( P \) has moved \( x \) cm from \( A \) is \( y \) cm\(^2\), express \( y \) in terms of \( x \) in the two following cases.

1. When \( P \) is moving along \( AB \)

2. When \( P \) is moving along \( BC \)

19 Find the following probabilities.

1. Two dice are thrown. Find the probability that their sum is not more than 5.

2. There are five people, \( a, b, c, d, \) and \( e \) divided into a group of two and a group of three. Find the probability that \( a \) is in the group of two.

3. A coin is tossed three times in a row. Find the probability that it lands the same way all three times.
20 The figure on the right is part of a larger figure. Complete this figure in the following two cases.

1. The figure has linear symmetry about line \( \ell \).
2. The figure has point symmetry about point \( O \).

21 Answer the following questions.

1. Construct the midpoint of line segment \( AB \).

2. Construct a line segment showing the distance between point \( A \) and line segment \( \ell \).

22 The diagram on the right shows a quadrilateral prism with a trapezoidal base. For each of the four following conditions, give all the answers that fit.

1. Edges parallel to edge \( BF \)
2. Faces perpendicular to edge \( AE \)
3. Faces perpendicular to face \( BFGC \)
4. Edges in a skew position with the edge \( BF \)?

23 In the diagram on the right, rotate the right triangle \( ABC \) about side \( AC \) as axis to construct a solid.

1. Draw a perspective view of the resulting solid.
2. Find the surface area and volume of the resulting solid.
24 In each of the following diagrams, find the size of $\angle x$.

\begin{align*}
1 & \ell \parallel m \\
2 & \ell \parallel m \\
3 & 30^\circ \\
4 & 135^\circ \\
5 & 80^\circ \quad 70^\circ \\
6 & 56^\circ \quad 62^\circ
\end{align*}

25 In $\triangle ABC$, with $\angle A = 90^\circ$ degrees, take a point D on BC such that $AB = AD$. Draw a perpendicular to AD through D, and let its point of intersection with AC be E. In this case, $\triangle EDC$ is isosceles. Prove this.

26 In $\triangle ABC$, let the bisector of $\angle A$ intersect BC at point D. Let a line segment through D parallel to AC intersect AB at point E. Next let a line segment through E parallel to BC intersect AC at point F. Now prove that $AE = CF$. 
1. Calculate the following.
   1) $\sqrt{5} \times \sqrt{15}$
   2) $\sqrt{35} \times \sqrt{4}$
   3) $\sqrt{10} \times \sqrt{8}$
   4) $\sqrt{42} \div \sqrt{6}$
   5) $\sqrt{45} \div \sqrt{5}$
   6) $\sqrt{72} \div \sqrt{12}$

2. Calculate the following.
   1) $\sqrt{6} + 3\sqrt{6}$
   2) $-5\sqrt{3} + 8\sqrt{3} - \sqrt{3}$
   3) $3\sqrt{5} + 2\sqrt{10} + 2\sqrt{5} - 4\sqrt{10}$
   4) $\sqrt{98} - 2\sqrt{18}$
   5) $\sqrt{75} + 7\sqrt{3} - 3\sqrt{48}$
   6) $2\sqrt{6} - \sqrt{18} + 3\sqrt{32} + \sqrt{54}$
   7) $\sqrt{40} + \frac{\sqrt{7}}{\sqrt{5}}$
   8) $\frac{21}{\sqrt{7}} - \sqrt{175}$

3. Calculate the following.
   1) $\sqrt{6} (\sqrt{3} - \sqrt{2})$
   2) $(2 + \sqrt{27}) \times \sqrt{12}$

Let's try!

A set of lockers is numbered from 1 to 100, and all the doors are closed. Now we carry out the following procedure.

1. For each locker whose number is a multiple of 1 (that is, all lockers), we open the door.
2. For each locker whose number is a multiple of 2, we close the door.
3. For each locker whose number is a multiple of 3, if the door is open we close it, and if it is closed we open it.
4. Continue this process up to multiples of 100.

At the end, what are the numbers of the lockers whose doors are open?
1. Calculate the following.
   1. \(-2x(4x-3)\)
   2. \((9a^2b + 21ab^2) \div \frac{3}{4}ab\)

2. Expand the following expressions.
   1. \((a-5)(a-8)\)
   2. \((a-1)(a+9)\)
   3. \((x-7)(x+3)\)
   4. \((a+4)^2\)
   5. \((x+9)(x-9)\)
   6. \((3a-2)^2\)
   7. \((2x-3)(2x+5)\)
   8. \((x+6)(2x-7)\)

3. Calculate the following.
   1. \(x(x-1) + (x-2)^2\)
   2. \((a+7)(a-7) - (a+2)(a+6)\)

4. Factorize the following expressions.
   1. \(6x^2 - 9x\)
   2. \(8a^2b + 2ab^2\)
   3. \(x^2 + 7x - 44\)
   4. \(x^2 - 15x + 36\)
   5. \(a^2 - 9a - 22\)
   6. \(x^2 - 16xy + 64y^2\)
   7. \(x^2 - 1\)
   8. \(9a^2 - 16b^2\)
   9. \(3x^2 + 18x + 15\)
   10. \(4x^2 + 16x + 16\)

Let's try!

Sacks of goods are piled in layers, and the number of sacks in each line is less than that in the line below by one. If the bottom layer has 18 sacks and the top layer has 8 sacks, how many sacks are there in total?

This problem also appeared in the Edo-period text *Jinkoki*.
PRACTICING SOLVING QUADRATIC EQUATIONS

1. Solve the following equations.
   1. \((x - 2)(x + 9) = 0\)
   2. \(x(x - 3) = 0\)
   3. \(x^2 + 7x = 0\)
   4. \(3x^2 = 4x\)
   5. \(x^2 - 2x - 3 = 0\)
   6. \(x^2 - 10x + 21 = 0\)
   7. \(x^2 + x - 56 = 0\)
   8. \(x^2 + 15x + 54 = 0\)
   9. \(x^2 + 3x - 28 = 0\)
   10. \(x^2 - 8x - 48 = 0\)
   11. \(x^2 + 2x + 1 = 0\)
   12. \(x^2 - 18x + 81 = 0\)
   13. \(x^2 - 64 = 0\)
   14. \(4x^2 - 9 = 0\)

2. Solve the following equations.
   1. \(x^2 - 32 = 0\)
   2. \(5x^2 = 45\)
   3. \(16x^2 - 7 = 0\)
   4. \(9x^2 - 50 = 0\)
   5. \((x + 4)^2 = 27\)
   6. \((x - 5)^2 - 16 = 0\)

3. Solve the following equations.
   1. \(x^2 - 7x + 3 = -2x + 9\)
   2. \(x(x + 6) = -9\)
   3. \((x - 3)(x - 4) = 2\)
   4. \((x + 6)(x - 6) = 5x\)
   5. \((x + 7)(x - 9) = -15\)
   6. \((x - 1)^2 = 2(x - 1)\)

Let's try!

For each of the two following questions, the blue boxes can be filled with a set of consecutive natural numbers.

Find the set of numbers in each case.

1. 365 = + + + +
2. 365 = \(z^2 + z^2 + z^2\)

190
Problems Involving Parabolas and Straight Lines

1. In the diagram on the right, (1) is the graph of a function \( y = ax^2 \) passing through the point (-2, 2).
(2) is the graph of the function \( y = -\frac{1}{2}x - 3 \).
Consider a point \( P (t, 0) \) on the \( x \) axis with a positive \( x \) coordinate. Let the points of intersection of a line through point \( P \) parallel to the \( y \) axis with the graphs (1) and (2) be \( A \) and \( B \) respectively.

1. Find the value of \( a \).
2. Find the value of \( t \) such that \( OA = OB \).

2. In the diagram on the right, the curve is the graph of the function \( y = \frac{1}{2}x^2 \), which intersects line \( m \) parallel to the \( x \) axis at points \( A \) and \( B \). \( P \) is a point on the \( x \) axis, with an \( x \) coordinate of 6.

1. If the equation for line \( m \) is \( y = 8 \), find the area of \( \triangle APB \).
2. \( Q \) is a point other than \( A \) on the line segment \( OA \). Find the equation of the straight line \( m \) when the areas of \( \triangle APB \) and \( \triangle QPB \) are equal.

Let's try!

The Edo-period mathematics text *Sanpo Keiko Zue* includes the following problem. Find the answer, taking 2 as approximately 1000.

We have one bean-paste cake on the first day, two on the second day, four on the third day, and so on, doubling the number of cakes each day. How many will there be on day thirty?
1. In each of the following diagrams, find the values of $x$ and $y$.
   1. $DE \parallel BC$
   2. $DE \parallel BC$
   3. $\angle BAC = \angle BCD$

2. In each of the following diagrams, $\ell$, $m$, and $n$ are all parallel. Find the value of $x$.
   1. $\ell$
   2. $\ell$
   3. $\ell$

3. In each of the following diagrams, find the value of $x$.
   1. $AB$, $CD$, and $EF$ are all parallel.
   2. Quadrilateral $ABCD$ is a parallelogram.

---

**Let's try!**

Six congruent squares are arranged inside a square of side 10 cm, as shown in the diagram on the right. Find the value of $x$. 

192
1. Quadrilateral ABCD in the diagram on the right is a parallelogram, with:
   \( AB = 6 \text{ cm}, \ BC = 4 \text{ cm}, \ \angle B = 120^\circ \)
   
   ① Find the area of parallelogram ABCD.

   ② Find the length of line segment AC.

2. The solid shown in the diagram on the right is a cube with an edge length of 6 cm, and M and N are the midpoints of edges BF and DH respectively.
   
   ① Find the length of line segment BN.

   ② Find the perimeter and area of the quadrilateral with vertices at A, M, G, and N.

Let's try!

Diagrams a and b on the right each consist of two concentric circles. AB and CD are chords of the outer circles which touch the inner circles, and \( AB = CD \).

In this case, which of the annular areas between the two circles is larger?
More Problems

1. Each sheet of a newspaper has four page numbers printed on it. If a newspaper consists of three sheets, it has a total of twelve pages, and when opened at the center spread the page numbers are as shown in the diagram on the right (for a Japanese newspaper opening from the right).

I take one sheet from a newspaper consisting of nine sheets, and this sheet includes page 13. Find the other page numbers on this sheet.

2. In a multiple-choice question with three answers a, b, and c, Alice, Bettie, and Carol replied as shown in the following table. The number in the last column is the number of correct replies each person gave.

From the information in this table, find the correct answers to the four questions.

<table>
<thead>
<tr>
<th></th>
<th>Problem 1</th>
<th>Problem 2</th>
<th>Problem 3</th>
<th>Problem 4</th>
<th>Score</th>
</tr>
</thead>
<tbody>
<tr>
<td>Alice</td>
<td>a</td>
<td>a</td>
<td>b</td>
<td>c</td>
<td>2</td>
</tr>
<tr>
<td>Bettie</td>
<td>a</td>
<td>c</td>
<td>a</td>
<td>b</td>
<td>3</td>
</tr>
<tr>
<td>Carol</td>
<td>b</td>
<td>c</td>
<td>b</td>
<td>a</td>
<td>2</td>
</tr>
</tbody>
</table>

Let's try!

In the "ladders" lottery shown on the right, A is assigned to 2, and C to 3. By adding one horizontal line, change this to assign A to 3 and C to 2.

There are two ways!
### Answers to chapter summary problems

#### Chapter 1 Square Roots

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3 &lt; \sqrt{10}</td>
</tr>
<tr>
<td>2</td>
<td>-3.88 &lt; -6 &lt; -3.33</td>
</tr>
<tr>
<td>2</td>
<td>\frac{2\sqrt{7}}{7}</td>
</tr>
<tr>
<td>3</td>
<td>Incorrect</td>
</tr>
<tr>
<td>4</td>
<td>14 \cdot 2</td>
</tr>
<tr>
<td>5</td>
<td>5.71 cm</td>
</tr>
</tbody>
</table>

#### Chapter 2 Polynomials

<table>
<thead>
<tr>
<th>Problem</th>
<th>Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2a^2 - 4ab</td>
</tr>
<tr>
<td>2</td>
<td>x^2 - 9x - 20</td>
</tr>
<tr>
<td>3</td>
<td>9a^2 + 6a + 1</td>
</tr>
<tr>
<td>4</td>
<td>23 - 4 \cdot 15</td>
</tr>
<tr>
<td>5</td>
<td>2m m(2m + 1)</td>
</tr>
<tr>
<td>6</td>
<td>Let ( n ) be an integer; then two consecutive integers can be represented as ( n ) and ( n - 1 ). This is equal to the sum of the two original numbers. Answer 195</td>
</tr>
</tbody>
</table>
Chapter 3 Quadratic Equations

1. \( x = \pm 1 \)
2. \( x = 8, \quad x = -\frac{3}{2} \)
3. \( x = 2 \pm \sqrt{7} \)
4. \( x = 0, \quad x = 15 \)
5. \( x = -2, \quad x = 6 \)
6. \( x = 6 \)
7. \( x = -9, \quad x = -7 \)
8. \( x = \pm 5 \)
9. \( x = 4, \quad x = 8 \)
10. \( x = -9, \quad x = 3 \)

Chapter 4 The Function \( y = ax^2 \)

1. \( a = \frac{1}{8} \)
2. \( a = -\frac{1}{2} \)
3. \( a = \frac{2}{3} \)
4. \( a = 2 \)
5. Maximum 11, minimum -4
6. Maximum 18, minimum 0
7. \( a = 1 \)
8. \( y = x^2 \)
9. \( y = 9 \)
10. Range of \( x: 0 \leq x \leq 4 \)
11. Range of \( y: 0 \leq y \leq 16 \)

Chapter 5 Similar Figures

1. \( x = 2 \)
2. \( x = 9 \)
3. Triangles similar to \( \triangle ACD: \triangle APE, \triangle CDP \)
4. \( \triangle DEC, \triangle EAD, \triangle PAC \)
5. \( \triangle BCA \)
3. In $\triangle ADP$ and $\triangle CBP$
   since inscribed angles for the arc $\overarc{AB}$ are equal
   \[ \angle PAD = \angle PCB \] \[ (1) \]
   \[ \angle P \text{ is common} \] \[ (2) \]
   From (1) and (2), since two pairs of angles are equal,
   \[ \triangle ADP \cong \triangle CBP \]

4. In $\triangle DAB$
   \[ PS \parallel AB, \ PS = \frac{1}{2} AB \]
   In $\triangle CAB$
   \[ RQ \parallel AB, \ RQ = \frac{1}{2} AB \]
   Therefore \[ PS \parallel RQ, \ PS = RQ \] 
   Since a pair of opposite sides is parallel and equal, the quadrilateral PSQR is a parallelogram.

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Chapter summary problems

1. In $\triangle ABC$ and $\triangle EDC$
   Since BC is a diameter
   \[ \angle BAC = \angle DEC = 90^\circ \] \[ (1) \]
   \[ \angle ABD = \frac{1}{2} \angle ABC = 30^\circ, \] and since
   inscribed angles for arc $\overarc{AB}$ are equal
   \[ \angle ECD = \angle ABD = 30^\circ \] \[ (2) \]
   But
   \[ \angle ACB = 180^\circ - \angle BAC - \angle ABC \]
   \[ = 30^\circ \] \[ (3) \]
   From (2) and (3), \[ \angle ACB = \angle ECD \] \[ (4) \]
   From (1) and (4), since two pairs of angles are equal
   \[ \triangle ABC \cong \triangle EDC \]

2. \( \text{If } AD = \frac{16}{5} \text{ cm, } BD = \frac{9}{5} \text{ cm} \)
   \[ CD = \frac{12}{5} \text{ cm} \]
   \[ 16 : 9 \]

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Chapter 6 The Pythagorean Theorem

Chapter summary problems

1. \( 8 \sqrt{2} \text{ cm}^2 \)
2. \( 3 \sqrt{5} \)
3. Height \( 3 \sqrt{3} \text{ cm}, \) volume \( \frac{125 \sqrt{3}}{3} \pi \text{ cm}^3 \)
4. \( \text{If } AB = \frac{20 \sqrt{3}}{3} \text{ cm} \)
   \[ BC = \frac{10 \sqrt{3}}{3} \text{ cm} \]
   \[ AD = 5 \sqrt{2} \text{ cm} \]
   \[ CD = 5 \sqrt{2} \text{ cm} \]
   \[ \text{If the length of AC is } a, \text{ for each } \]
   \[ \text{of the right triangles the height is } \frac{a}{2}, \text{ and they are equal.} \]

5. The relation \( p + q = r \) holds.

---

Chapter summary problems

1. \( 29 \text{ cm} \)
2. \( \text{Let } BC = a, \ CA = b, \ AB = c \)
   The area of the shaded portions is:
   \[ \frac{1}{2} \pi \left( \frac{1}{2} b \right)^2 + \frac{1}{2} \pi \left( \frac{1}{2} c \right)^2 \]
   \[ + \frac{1}{2} b \times c \]
   \[ = \frac{1}{8} \pi (b^2 + c^2 - a^2) + \frac{1}{2} b c \]
   Since \( a^2 = b^2 + c^2, \) the area of the shaded portions is \( \frac{1}{2} b c, \) which is equal to the area of the right triangle ABC.

3. \( \text{If } \text{ in } \triangle ABH \)
   \[ a^2 = 169 - x^2 \]
   \[ \text{In } \triangle ACH \]
   \[ a^2 = -x^2 + 28x + 29 \]

   \[ x = 5 \]

3. 84

Answer 197


**Answers to Development Questions**

Numbers Which Cannot Be Expressed as Fractions p.26-27

**Problem 1**

\[
0.4 = \frac{4}{9}, \quad 0.123 = \frac{41}{333}
\]

Eliminating Root Signs from the Denominator of an Expression p.40

**Problem 1**

1. \(\sqrt{5} - \frac{1}{2}\)
2. 0.618

Using Algebraic Variables p.54

**Problem 1**

1. \(x^2 + 2xy + y^2 - 2x - 2y = 15\)
2. \(a^2 + 2ab + b^2 + 2ac + 2bc + c^2\)
3. \(a^2 - 2ab + b^2 - 12a + 12b + 36\)
4. \(a^2 - b^2 + 8b - 16\)

**Problem 2**

1. \((a + b + 2)(a + b + 3)\)
2. \((a - 2)(a - 9)\)
3. \((3x + 4)(x + 10)\)
4. \((a - b)(a - y)\)

Quadratic Equations with an Odd Coefficient of \(x\) p.65

**Problem 1**

1. 25, 5
2. 9, 3

**Problem 2**

1. \(x = -2 \pm 2\sqrt{2}\)
2. \(x = \frac{5 \pm 17}{2}\)

Quadratic Equations of the Form \(x^2 + px + q = 0\) p.73-75

**Problem 1**

1. \(x = -\frac{5 \pm \sqrt{33}}{2}\)
2. \(x = 4 \pm 2\sqrt{3}\)

**Problem 2**

1. \(x = -7, \quad x = 3\)
2. \(x = 4\)

198 Answer

Instantaneous Speed p.92

Elapsed time from 1 s to 1.01 s ... 2.01 m/s
Elapsed time from 1 s to 1.001 s ... 2.001 m/s
Elapsed time from 1 s to 1.0001 s ... 2.0001 m/s

**Problem 1**

8 m/s

Finding the Catch-up Point p.152-154

**Problem 1**

Boris will arrive first.

**Problem 2**

\(y = 4x + 4\)

**Problem 3**

\((x - 4)^2 = 24\)

\(x - 4 = 4\sqrt{3}\)

\(x = 4 + 2\sqrt{3}\)

Since \(x > 0\),

\(x = 4 + 2\sqrt{3}\)

Substituting \(4 + 2\sqrt{3}\) for \(x\) in \(y = 4x + 4\) we get

\(y = 4(4 + 2\sqrt{3}) + 4\)

\(= 20 + 8\sqrt{3}\)

He catches up \((4 + 2\sqrt{3})\) seconds after starting, at a point \((20 + 8\sqrt{3})\) meters from P.

Let's try:

\((4, 16), \quad (-1, 1)\)

**Problem 4**

1. \(y = 4x - 6\)
2. Alex is ahead for the interval between 2 seconds and 6 seconds after Boris sets off
Problem 5

(1) $y = 4x - 8$

(2) At four seconds after Boris starts out, Alex instantaneously catches up with Boris, but immediately falls behind. Therefore Alex will not be able to pass Boris.

Problem 6

(1) Omitted

(2) No, he will not catch up.

(3) There are no solutions to the equations.

Using Similar Figures to Investigate Properties of Circles

Page 155–157

Problem 1

PA × PB, PC × PD

Problem 2

$\Delta ADP \sim \Delta CBP$ holds.

Therefore, corresponding sides of the two triangles are in the same ratio, and thus

PA : PC = PD : PB

In other words,

PA × PB = PC × PD

Problem 3

PA × PB, PC × PD

Problem 4

PQ × PR = PA × PB

Since PQ = PR, PA = 2, PB = 3,

$PQ^2 = 2 \times 3$

$PQ = \sqrt{6}$

Problem 5

For example, since $1 \times 6 = 6$, you can use a circle of diameter 7 for the construction.

Problem 6

Construct a circle whose diameter is the sum of the length and width of the rectangle.

Area and Volume of Similar Figures — p.158–161

Problem 1

Omitted

Problem 2

(1) $B'C' = 10 \text{ cm}$

$A'T' = 8 \text{ cm}$

(2) Area of $\triangle ABC$: 10 cm²

Area of $\triangle A'B'C'$: 40 cm²

(3) $1:4$

Problem 3

Ratio of the circumferences: 3:5

Ratio of the areas: 9:25

Problem 4

(1) $3:4$

(2) $64 \text{ cm}^2$

Problem 5

$b: 3d$, $c: 5d$, $d: 7d$

Problem 6

$\sqrt{2} \text{ cm}$

Problem 7

Ratio of the surface areas of P and Q: 4:9

Ratio of the volumes of P and Q: 8:27

Problem 8

(1) $120 \pi \text{ cm}^3$

(2) $2:5$

(3) $\frac{192 \pi}{25} \text{ cm}^3$

Let's try!

7.94 cm

Joining the Midpoints — p.162–163

As shown in the diagram on the right, the triangles get progressively smaller, and approach a single point.

Answer 199
Problem 1
1 By the triangle midpoint theorem,
   \[ AC_1 = B_1A_1, \quad AC_1 \neq B_1A_1 \]
   Since a pair of opposite sides are parallel and equal, the quadrilateral
   \( AC_1B_1A_1 \) is a parallelogram.

2 The diagonals of a parallelogram intersect at their midpoints. \( A_2 \) is the
   midpoint of \( C_1B_1 \). Therefore, \( A_2 \) is the intersection of the
   diagonals \( AA_1 \) and \( B_1C_1 \).

Problem 2
Since \( A_1 \) and \( B_1 \) are the midpoints of \( CB \) and \( CA \) respectively, by the triangle
midpoint theorem,
   \[ A_1B_1 \neq BA, \quad A_1B_1 = \frac{1}{2} BA \]
Therefore by the theorem on triangles and ratios (1)
   \[ GA : GA_1 = 2 : 1 \]

Problem 3
Omitted

Problem 4
\( \triangle ABC \) and \( \triangle BCG \), taking side \( BC \) as the base, the heights are in the ratio 3:1. Therefore:
   \[ \triangle BCG = \frac{1}{3} \triangle ABC \]
   And similarly for \( \triangle ABG \) and \( \triangle CAG \).

Review Problems

----- Numerical Calculation ----- p.182
1  1) -8  2) -13
   3) 11  4) -2
   5) -14  6) 13
   7) -7  8) 0

2  1) -42  2) 6
   3) -21  4) 50
   5) -84  6) -9
   7) 31  8) -21

3  1) \(-1 < 0 < 0.5 \)
   2) -2, -1, 0, 1, 2
   3) \( a \) is negative; \( b \) is positive

4  1) 164 cm
   2) 12 cm taller
   3) 169 cm

----- Algebraic Calculation ----- p.183
5  1) \((3x + 2y) \) kg
   2) \((1000 - 10a) \) yen

6  1) \(-a + 3 \)
   2) \(4x + 2y \)
   3) \(10x - 1 \)
   4) \(8a + b \)
   5) \(8x + 12y \)
   6) \(5a - 2b \)
   7) \(4a - 15b \)
   8) \(\frac{14x - 11}{15} \)
   9) \(-14xy \)
   10) \(15a \)

7  1) 7
   2) -2
   3) -6
   4) 20

8  1) \(b = -a + \frac{f}{2} \)
   2) \(h = \frac{3V}{S} \)

---

Slope and Height of a Roof — p.164~165

Approximately 0.6 m
In order, approximately 0.18, ...

Problem 1
In order, approximately 1.3 m, ...

Problem 2
Approximately 60 degrees

Problem 3
Approximately 220 m
For four consecutive integers such that the smallest is a multiple of 3, then if \( n \) is an integer, the four integers can be represented as follows.

\[ 3n, \ 3n+1, \ 3n+2, \ 3n+3 \]

The sum of these is

\[ 3n + (3n+1) + (3n+2) + (3n+3) = 6(n+1) \]

Since \( 2n+1 \) is an integer, the sum of the four numbers is a multiple of 6.

Omitted

Equations and Simultaneous Equations

11. \( \begin{align*} \frac{4}{5}, \ c \end{align*} \)

12. \( \begin{align*} x &= \frac{1}{2} & x &= 3 \\ x &= -2 & x &= 0 \\ x &= 2 & x &= 4 \end{align*} \)

13. \( \begin{align*} x &= 2, \ y &= 1 \\ x &= -3, \ y &= 5 \\ x &= 8, \ y &= -1 \\ x &= -2, \ y &= -6 \end{align*} \)

14. \( a = 4 \)

15. \( 58 \)

16. \( 1200 \text{ m} \)

Functions and Probability

17. \( \begin{align*} y &= -\frac{3}{2}x \\ y &= -8 \\ y &= -3x - 1 \\ 2 \leq y \leq 8 \end{align*} \)

18. \( \begin{align*} y &= 2x \\ y &= \frac{3}{2}x + 2 \end{align*} \)

19. \( \begin{align*} \frac{5}{18} \\ \frac{2}{5} \\ \frac{1}{4} \end{align*} \)

20. \( \begin{align*} \text{Edges AE, CG, and DH} \\ \text{Faces ABCD and EFGH} \\ \text{Faces ABCD, AEFB, EFGH, and DHGC} \\ \text{Edges AD, CD, EH, and GH} \end{align*} \)

21. \( \begin{align*} \text{Volume: } 96\pi \text{ cm}^3 \end{align*} \)

22. \( \begin{align*} \text{Surface area: } 96\pi \text{ cm}^2 \end{align*} \)

23. \( \begin{align*} \text{Volume: } 96\pi \text{ cm}^3 \end{align*} \)
**Geometrical Properties** p.187

24. (1) 45° (2) 70° (3) 24° (4) 70° (5) 25° (6) 55°

25. In \( \triangle EDC \),

\[ \angle ECD = 180° - \angle BAC - \angle B \]
\[ = 90° - \angle B \] ........................ (1)

\[ \angle EDC = 180° - \angle ADE - \angle ADB \]
\[ = 90° - \angle ADB \] ........................ (2)

Since \( AB = AD \), \( \triangle ABD \) is an isosceles triangle. Therefore:

\[ \angle B = \angle ADB \] ........................ (3)

From (1), (2), and (3)

\[ \angle ECD = \angle EDC \]

Since two angles are equal, \( \triangle EDC \) is isosceles.

26. In \( \triangle EAD \),

By hypothesis \( \angle EAD = \angle CAD \) ... (1)

\( ED \parallel AC \) and alternate angles are equal, thus

\[ \angle EDA = \angle CAD \] ........................ (2)

From (1) and (2)

\[ \angle EAD = \angle EDA \]

Since two angles are equal, \( \triangle EAD \) is isosceles. Therefore

\[ AE = DE \] ........................ (3)

In the quadrilateral \( EDCF \),

By hypothesis \( ED \parallel AC, EF \parallel BC \)

Since two pairs of opposite sides are parallel, the quadrilateral \( EDCF \) is a parallelogram.

Therefore

\[ DE = CF \] ........................ (4)

From (3) and (4)

\[ AE = CF \]

Answers to Supplementary Problems

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**Practicing Calculation of Expressions Including Root Signs** p.188

1. (1) \( 5\sqrt{3} \) (2) \( 7, 10 \) (3) \( 4\sqrt{5} \)

2. (1) \( 4\sqrt{6} \) (2) \( 2\sqrt{3} \)

3. (1) \( \sqrt{2} \) (2) \( 9\sqrt{2} + 5\sqrt{6} \)

4. (1) \( 3\sqrt{2} \) (2) \( 2\sqrt{3} \)

**Practicing Expanding Expressions and Factorization** p.189

1. (1) \( -8x^2 + 6x \) (2) \( 12a + 28b \)

2. (1) \( a^2 + 3a + 10 \) (2) \( a^2 + 8a + 9 \)

3. (1) \( x^2 - 4x - 21 \) (2) \( a^2 + 8a + 16 \)

4. (1) \( x^2 - 81 \) (2) \( 9a^2 - 12a + 4 \)

5. (1) \( 4x^2 + 4x - 15 \) (2) \( 2x^2 + 5x - 42 \)

6. (1) \( 2x^2 - 5x - 4 \) (2) \( -8a - 61 \)

7. (1) \( 3x(2x - 3) \) (2) \( 2a(4a + b) \)

8. (1) \( (x - 4)(x + 11) \) (2) \( (x - 3)(x - 12) \)

9. (1) \( (a - 2)(a - 11) \) (2) \( (x - 8y)^2 \)

10. (1) \( (x + 1)(x - 1) \) (2) \( (3a + 4b)(3a - 4b) \)
Practicing Solving Quadratic Equations — p. 190

1. $x = 2, x = -9$
2. $x = 0, x = 3$
3. $x = 0, x = 7$
4. $x = 0, x = \frac{4}{3}$
5. $x = -1, x = 3$
6. $x = 3, x = 7$
7. $x = -8, x = 7$
8. $x = -9, x = -6$
9. $x = -7, x = 4$
10. $x = -4, x = 12$
11. $x = 1$
12. $x = 9$
13. $x = \pm 8$
14. $x = \pm \frac{3}{2}$

2. $x = \pm 4\sqrt{2}$
   $x = \pm 3$
   $x = \pm \sqrt{7}$
   $x = \pm \frac{5\sqrt{2}}{3}$
   $x = -4 \pm 3, \frac{3}{3}$
   $x = 9, x = 1$

3. $x = -1, x = 6$
   $x = -3$
   $x = 2, x = 5$
   $x = -4, x = 9$
   $x = -6, x = 8$
   $x = 1, x = 3$

Problems Involving Parabolas and Straight Lines — p. 191

1. \(a = \frac{1}{2}\)
2. \(l = 3\)

2. \(y = \frac{9}{2}\)

Practicing Finding Line Segment Lengths — p. 192

1. $x = 5, y = 18$
2. $x = 8, y = 4$
3. $x = 15, y = 9$

2. $x = 4$
   $x = 9$
   $x = 4$

3. $x = \frac{18}{3}$
   $x = \frac{24}{5}$

Finding Line Segment Lengths and Areas — p. 193

1. $12 \sqrt{3} \text{ cm}^2$
2. $2\sqrt{19} \text{ cm}$

2. $9 \text{ cm}$
   $12 \sqrt{5} \text{ cm}$
   $18 \sqrt{6} \text{ cm}^2$

More Problems — p. 194

1. 14, 23, 24

2. Problem 1:a  Problem 2:c
   Problem 3:b  Problem 4:b
Summary of terminology used in the first and second year

**Calculation with numbers**
- Negative numbers, minus sign
  
  \(-1, -2, -3, -4, \ldots\)
- Positive numbers, plus sign
  
  \(+1, +2, +3, +4, \ldots\)
- Natural numbers
  
  \(1, 2, 3, 4, 5, \ldots\)
- Origin
- Positive direction, negative direction
- Absolute value
- Absolute value \(+3\) or \(-3\) is \(\)\]
- Inequality signs \(<, >, \leq, \geq\)
- Addition, subtraction, multiplication, and division
- Commutative law of addition, associative law of addition
  
  \[a + b = b + a\]
  
  \[(a + b) + c = a + (b + c)\]
- Commutative law of multiplication, associative law of multiplication
  
  \[a \times b = b \times a\]
  
  \[(a \times b) \times c = a \times (b \times c)\]
- Distributive law
  
  \[a \times (b + c) = a \times b + a \times c\]
- Power, exponent
  
  \[2 \times 2 \times 2 \times 2 \times 2 = 2\]
- Second power, square, third power, cube
  
  \[3^2, a^3, 5^2, b^3\]
- Reciprocal
  
  The reciprocal of 4 is \([\]\]
- Term
  
  \([\pi]\)

**Algebra**
- Substitution and evaluation
- Term
  
  The terms in \(2x - 5y\) are \([\]\) and \([\]\]
- Coefficient
  
  The coefficient of \(-3x\) is \([\]\]
- Monomial, polynomial
- Terms (in a polynomial)
- Degree (of a monomial or polynomial)
  
  The degree of \(2x^3 + 4x^2 - 5x\) is \([\]\]
- Similar terms
- Linear expression

**Equations and simultaneous equations**
- Equation, left-hand side (LHS), right-hand side (RHS)
  
  \[\text{Equation} \quad 2x + 3 = 4x - 1\]
  
  \[\text{LHS} \quad \text{RHS}\]
- Equation
- Linear equation, root, solving
- Properties of equations
  
  If \(A = B\) then \(A + C = B + C\)
  
  If \(A = B\) then \(A - C = B - C\)
  
  If \(A = B\) then \(AC = BC\)
  
  If \(A = B\) then \(A/C = B/C\)
- Rearranging terms
- Canceling the denominator
- Linear equation in two variables
- Simultaneous equation, root, solving
- Cancellation
- Method of addition and subtraction, substitution method
Proportionality and inverse proportionality: linear functions

- Variable, range
- Proportionality
  \[ y = ax \]
- Constant of proportionality
- \( x \)-axis, \( y \)-axis, coordinate axes, origin
- \( x \)-coordinate, \( y \)-coordinate, coordinates
- Inverse proportionality
  \[ y = \frac{a}{x} \]
- Hyperbola
- Function
- Linear function
  \[ y = ax + b \]
- Rate of change
- Intercept, slope
- Graph of an equation

Plane geometry

- Line symmetry, axis of symmetry
- Point symmetry, center of symmetry
- Line segment, straight line, half line
- Line segment
  - Straight line
  - Half line
- \( \triangle AOB \)
- \( AB \parallel CD \)
- Midpoint
- Arc, chord
- Sector, central angle
• Polygon, regular polygon
• \( \triangle ABC, \square ABCD \)
• Perpendicular
• Perpendicular bisector, angle bisector
• Tangent, point of contact
• Interior angle, exterior angle
• Vertically opposite angles
• Corresponding angles, alternate angles
• Apex, base, base vertex
• Hypotenuse
• Opposite sides, opposite angles
• Inscribed angle
• Proof
• Hypothesis, conclusion
• Definition, theorem
• Converse
• Congruence (\(\cong\))

**Solid geometry**
• Polyhedron, regular polyhedron
• Prism, cylinder, pyramid, cone
• Skew position
• Generator, solid of revolution
• Surface area, lateral surface area, base area

**Probability**
• Probability
• Equally likely outcomes
• Tree diagram
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Wasan Topics

In the Edo period some highly original contributions to mathematics were made in Japan, and the term *wasan* is used for these. The book *Jinkoki* introduced earlier is an important text from this period; let's look at some of the topics it contains.

Names for large numbers

Names are listed for numbers up to enormous powers of ten.

Nested pots

We have a set of nested pots*, whose capacities are, in increasing order: 1 sho, 2 sho, 3 sho, 4 sho, 5 sho, 6 sho, 7 sho, 8 sho. If the total cost of these pots is 43 momme and 2 bu, find the cost of the 1-sho pot.

* This refers to pots of different sizes, which can be fitted inside each other like Russian dolls.

The apothecary's problem

Go stones are arranged in a square, as shown by the black circles. When there are eight stones on each side, the stones are rearranged as shown by the white circles. From the fact that the number of stones in the last row is four, find the total number of stones.
Proving the Pythagorean theorem

Diagram: Step-by-step visual representation of the theorem's proof.
Geometry

Find the words or symbols to fit in the boxes.

- **Similarity**
  - Conditions for similar triangles → see page 109
  - Three pairs of □□□ are in the same ratio
    - ![Diagram](image1)
  - Two pairs of □□□ are in the same ratio, and the included □□□ are equal
    - ![Diagram](image2)
  - Two pairs of □□□ are equal
    - ![Diagram](image3)

- **The Pythagorean theorem**
  - Pythagorean theorem → see page 132
    - ![Diagram](image4)
  - Converse of the Pythagorean theorem → see page 135
    - ![Diagram](image5)

- **Ratios of triangles**
  - ![Diagram](image6)
  - ![Diagram](image7)
  - ![Diagram](image8)

- **Midpoint linkage theorem**
  - ![Diagram](image9)

- **Parallel lines and ratios**
  - ![Diagram](image10)
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